# VECTOR BUNDLES OVER ANALYTIC CHARACTER VARIETIES

TOBIAS SCHMIDT

ABSTRACT. Let $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ be a chain of complete intermediate fields where $\mathbb{Q}_p \subseteq L$ is finite and $K$ discretely valued. Let $Z$ be a one dimensional finitely generated abelian locally $L$-analytic group and let $\hat{Z}_K$ be its rigid $K$-analytic character group. Generalizing work of Lazard we compute the Picard group and the Grothendieck group of $\hat{Z}_K$. If $Z = o$, the integers in $L \neq \mathbb{Q}_p$, we find $Pic(\hat{o}_K) = \mathbb{Z}_p$ which answers a question raised by J. Teitelbaum.

## 1. Introduction

Let $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ be a chain of complete intermediate fields where $\mathbb{Q}_p \subseteq L$ is finite and $K$ is discretely valued. Let $\mathbb{G}$ be a quasi-split connected reductive group over $L$ and $G := \mathbb{G}(L)$ its group of $L$-valued points, viewed as a locally $L$-analytic group. The category $\mathrm{R}ep_K^a(G)$ of (essentially) admissible locally $L$-analytic $G$-representations over $K$ is of general interest in the realm of the $p$-adic Langlands programme. Let $\mathbb{B}$ be a Borel subgroup of $\mathbb{G}$, $\mathbb{N}$ its unipotent radical, $\mathbb{T} := \mathbb{B}/\mathbb{N}$ its Levi quotient and $T := \mathbb{T}(L)$ the group of $L$-valued points. The locally analytic Jacquet-functor associated to these data relates $\mathrm{R}ep_K^a(G)$ to the much simpler category $\mathrm{R}ep_K^a(T)$. In particular, the latter is known to be equivalent to the category of coherent module sheaves on the rigid character group $\hat{T}_K$ of $T$.

The purpose of this note is to shed some first light on the isomorphism classes of vector bundles over a space like $\hat{T}_K$. Let therefore $Z$ be a one dimensional topologically finitely generated abelian locally $L$-analytic group and denote by $\hat{Z}_K$ its rigid character group. Our main result (Thm. 5.14) computes the Picard group and the Grothendieck group of $\hat{Z}_K$. More precisely, denote by $\mu \subseteq Z$ the (finite) torsion subgroup of $Z$. We show that rank and determinant mappings induce an isomorphism

$$\mathrm{rk} \oplus \det : K_0(\hat{Z}_K) \xrightarrow{\cong} \mathbb{Z}^{\#\pi_0(\hat{Z}_K)} \oplus Pic(\hat{Z}_K)$$

where $\#\pi_0(\hat{Z}_K) \leq \#\mu$. Furthermore, $Pic(\hat{Z}_K) = 0$ if $L = \mathbb{Q}_p$ and $Pic(\hat{Z}_K) = \mathbb{Z}_p^{\#\pi_0(\hat{Z}_K)}$ canonically otherwise. In particular, if $Z = o$, the integers in $L \neq \mathbb{Q}_p$ we find $Pic(\hat{o}_K) = \mathbb{Z}_p$ where a topological generator is given by the ideal sheaf defining the zero section $\mathrm{Sp}\, K \to \hat{o}_K$. This answers a question raised by J. Teitelbaum (cf. [Tei]). On the other hand if $Z = \mathbb{Z}_p$ then $\hat{Z}_K = \mathbf{B}_K$, the open unit disc around zero, and our result reduces to a well-known theorem of M. Lazard (cf. [Laz62]).

We deduce the main result from a general structure result (Thm. 4.7) on the Picard group of twisted affinoid discs. More precisely, let $X$ be a $K$-affinoid twisted disc, trivialized by a finite extension $K \subseteq K'$. We show that $Pic(X)$ is finite cyclic of order dividing the $p$-part of the ramification index of $K'$ over $K$. Furthermore, one may deduce from $p$-adic Fourier theory (cf. [ST01]) that the space $\hat{o}_K$ is an increasing union over such twisted discs. Applying our result and passing to the limit gives the main result in case $Z = o$. The general case relies on the variety $\hat{Z}_K$ decomposing into the fibre product of a finite étale part, a toral part and $\hat{o}_K$.

We now briefly outline the paper. We begin by establishing some basic results on the $K_0$-theory of the types of rings we will encounter in the following. We proceed by collecting results on the $p$-radical descent theory of certain Krull domains in characteristic $p$. This is then applied to (Temkin) reductions of twisted Tate algebras and yields the structure of their Picard groups. The results are applied to the variety $\hat{o}_K$ and then to the general case $\hat{Z}_K$ yielding the main result. We finish by clarifying the relation of the groups $Pic(\hat{Z}_K)$ and $K_0(\hat{Z}_K)$ to the corresponding groups associated to the algebra of $K$-valued locally $L$-analytic distributions on $Z$. It follows that the (generalized) Amice Fourier homomorphism induces canonical isomorphisms between them.

*Acknowledgements.* The author would like to thank P. Schneider and J. Teitelbaum for interesting conversations. He is grateful to J. Kohlhaase for reading the manuscript so carefully.

## 2. Some algebraic $K_0$-theory

Throughout this section let $R$ be a commutative associative unital ring. We begin by recalling some basic notions from algebraic $K$-theory (cf. [Bas68]). Let $\mathbf{P}(R)$ be the abelian monoid consisting of isomorphism classes of finitely generated projective $R$-modules with binary operation given by the direct sum. The *Grothendieck group* $K_0(R)$ of $R$ is the group completion of $\mathbf{P}(R)$. The *Picard group* $Pic(R)$ of $R$ is the subset of $\mathbf{P}(R)$ consisting of classes of constant rank one modules with binary operation given by the tensor product. Both constructions induce covariant functors to abelian groups. Mapping $1 \mapsto [R]$ induces an injective group homomorphism $\mathbb{Z} \to K_0(R)$. If $[P] \in \mathbf{P}(R)$ is of constant rank $r$ then $\det([P]) := [\wedge^r P] \in Pic(R)$. This gives a surjective group homomorphism $\det : K_0(R) \to Pic(R)$, functorial in $R$ with $\mathbb{Z} \subseteq \ker \det$. Denoting by $H_0(R)$ the abelian group of continuous maps $Spec(R) \to \mathbb{Z}$ the rank of an $R$-module induces a homomorphism $\mathrm{rk} : K_0(R) \to H_0(R)$ which yields a surjection

$$\mathrm{rk} \oplus \det : K_0(R) \to H_0(R) \oplus Pic(R).$$

The kernel $SK_0(R)$ consists of classes $[P] - [R^m]$ where $P$ has, say, rank $m$ and $\wedge^m P \cong R$.

Recall that $R$ is called *semihereditary* resp. *coherent* resp. *regular* if any finitely generated ideal is projective resp. is finitely presented resp. has finite projective dimension. A semihereditary ring is coherent regular. For a concise account of the theory of commutative coherent rings we refer to [Gla89].

**Lemma 2.1.** *Let $R$ be semihereditary and let $R' := R[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$ be the Laurent polynomial ring over $R$ in the variables $t_1, ..., t_n$. The natural map $R \to R'$ induces an isomorphism on Picard and Grothendieck groups.*

*Proof:* This is a combination of well-known results. Since $R$ is semihereditary $R'$ is coherent ([loc.cit.], Cor. 7.3.4/Thm. 2.4.2.) Then $R'$ is regular by [loc.cit.], Thm. 8.2.14 applied to the abelian group $\mathbb{Z}^n$. Now $R'$ being coherent the category of finitely presented $R'$-modules is abelian and $R'$ being regular implies, according to [loc.cit.], Thm. 6.2.1 that any object in this category has a finite resolution by finite free modules. In this situation the result follows along the same lines as in the case of noetherian regular rings (cf. [MR87], Cor. 12.3.6). $\square$

Recall that an integral domain $R$ is called a *Prüferian domain* if any finitely generated ideal is invertible. A Prüferian domain is the same as a semihereditary domain and therefore coherent regular.

**Lemma 2.2.** *Let $R$ be noetherian of Krull dimension one or let $R$ be a Prüferian domain. Then $SK_0(R) = 0$.*

*Proof:* In both cases Serre's theorem (cf. [Bas68], Thm. IV.2.5) yields that any finitely generated projective module is isomorphic to a direct sum of an invertible and a free module (cf. also [Gru68], Remarque V.2.3). The explicit description of $SK_0(R)$ above then shows this group to be trivial. $\square$

Recall that $R$ is called *filtered* if it is equipped with a family $(F^sR)_{s\in\mathbb{R}}$ of additive subgroups $F^sR \subseteq R$ such that for $r, s \in \mathbb{R}$ we have $F^sR \supseteq F^rR$ if $s \leq r$, $F^sR \cdot F^rR \subseteq F^{s+r}R$ and $1 \in F^0R$. For $s \in \mathbb{R}$ let $F^{s+}R := \cup_{r>s} F^rR$ and $gr^sR := F^sR/F^{s+}R$. The ring $gr^\bullet R := \oplus_{s\in\mathbb{R}} gr^sR$ together with the obvious multiplication is called the *associated graded ring*. As usual, a *homomorphism of filtered rings* $\phi : R \to R'$ is a unital ring homomorphism respecting the filtrations and, thus, inducing a ring homomorphism $gr^\bullet R \to gr^\bullet R'$. The filtration on a filtered ring $R$ is called *quasi-integral* if there exists $n_0 \in \mathbb{N}$ such that $\{s \in \mathbb{R} : gr^sR \neq 0\} \subseteq \mathbb{Z} \cdot \frac{1}{n_0}$ (cf. [ST03], Sect. 1). In this case rescaling allows us to view $R$ as a $\mathbb{Z}$-filtered ring to which the usual notions of the theory of $\mathbb{Z}$-filtered rings apply (e.g. [LvO96]). In particular, denote by $R$-filt the category of $\mathbb{Z}$-filtered $R$-modules. Let $gr^\bullet R$-gr and $\tilde{R}$-gr be the categories of $\mathbb{Z}$-graded modules with graded morphisms of degree zero over the graded ring $gr^\bullet R$ and the *Rees* ring $\tilde{R}$ (cf. [loc.cit.], Def. I.4.3.5) of $R$ respectively. Finally, for our purposes, a $\mathbb{Z}$-filtered noetherian ring $R$ is called a *Zariski ring* if the filtration is exhaustive and complete and the ring $gr^\bullet R$ is noetherian (cf. [loc.cit.], Prop. II.2.2.1). For any Zariski ring $R$ with regular graded ring the $K_0$-part of Quillen's theorem (in the generalized version of [loc.cit.], Cor. II.6.6.8) yields an isomorphism

$$\nu : K_0(R) \xrightarrow{\cong} K_0(gr^\bullet R) \tag{1}$$

with $\nu[R] = [gr^\bullet R]$ fitting into the commutative diagram of abelian groups

$$\begin{array}{ccccccc}
K_{0g}(gr^\bullet R) & \xrightarrow{\alpha} & K_{0g}(\tilde{R}) & \xrightarrow{\beta} & K_0(R) & \longrightarrow & 0 \\
{\scriptstyle =}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle \nu}\downarrow & & \\
K_{0g}(gr^\bullet R) & \xrightarrow{\varphi} & K_{0g}(gr^\bullet R) & \xrightarrow{\gamma} & K_0(gr^\bullet R) & \longrightarrow & 0.
\end{array}$$

Here, $K_{0g}$ equals $K_0$ applied to the category of finitely generated projective graded modules over a $\mathbb{Z}$-graded ring with degree zero graded morphisms. The upper horizontal sequence is a version of the usual localization exact sequence (cf. [loc.cit.], Thm. 6.6.5) and the lower exact sequence comes from [VdB86]. More precisely, $\beta$ is induced from the localization $\tilde{R} \to \tilde{R}_{(X)}$ at the canonical regular central element $X \in \tilde{R}$ together with the isomorphism $\tilde{R}_{(X)} \cong R[t^{\pm 1}]$ ($t$ a variable) and the equivalence of categories $M \mapsto M \otimes_R R[t^{\pm 1}]$ between $R$-modules and $\mathbb{Z}$-graded $R[t^{\pm 1}]$-modules. The map $\gamma$ comes from the functor "forgetting the gradation". The mapping $i$ is induced from the canonical map $\tilde{R} \to \tilde{R}/X\tilde{R} \cong gr^\bullet R$ and $\nu$ is defined so as to make the diagram commute.

**Lemma 2.3.** *Let $R$ be a one dimensional integral Zariski ring with regular integral graded ring. The isomorphism $\nu$ induces an isomorphism $Pic(R) \xrightarrow{\cong} Pic(gr^\bullet R)$.*

*Proof:* Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & K_0(R) & \xrightarrow{\det} & Pic(R) & \longrightarrow & 0 \\
 & & {\scriptstyle =}\downarrow & & {\scriptstyle \nu}\downarrow & & {\scriptstyle \psi}\downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & K_0(gr^\bullet R) & \xrightarrow{\det} & Pic(gr^\bullet R) & \longrightarrow & 0
\end{array}$$

where the left-hand square commutes because of $\nu[R] = [gr^\bullet R]$. In both rows $\mathbb{Z} \subseteq \ker \det$ holds and, by Lemma 2.2, the upper horizontal row is exact whence a surjective homomorphism $\psi : Pic(R) \to Pic(gr^\bullet R)$ making the diagram commutative. Now reconsider the maps $i$, $\gamma$, $\beta$ above. On an isomorphism class of a module $i$, $\beta$, $\gamma$ and therefore $\nu$ preserve the rank. On the level of sets $\det|_{Pic(R)} = id$ (and similarly for $gr^\bullet R$) whence $\psi$ must be injective. $\square$

**Corollary 2.4.** *Let $R$ be a Zariski ring and $M$ a finitely generated $R$-module. If $M$ is a projective object in $R$-*filt *endowed with a good filtration then $gr^\bullet M$ is a finitely generated projective $gr^\bullet M$-module and $\nu([M]) = [gr^\bullet M]$.*

*Proof:* If $M$ is $R$-filt projective its graded module $gr^\bullet M$ and its Rees-module $\tilde{M}$ are projective objects in $gr^\bullet R$-gr and $\tilde{R}$-gr respectively (cf. [LvO96], Lem. I.6.4/Prop. I.6.5). Since the filtration on $M$ is good both graded modules are finitely generated (cf. [loc.cit.], Lem. I.5.4). Considering the diagram at the end of [loc.cit.], I.4.3 we find $\beta([\tilde{M}]) = [M]$. Hence, $\nu([M]) = \nu \circ \beta([\tilde{M}]) = \gamma \circ i([\tilde{M}]) = [gr^\bullet M]$. $\square$

## 3. $p$-radical descent of higher exponent

3.1. **Krull domains.** We recall some divisor theory of Krull domains (cf. [Bou98], VII.1) thereby fixing some notation. An integral domain $A$ is called a *Krull domain* if there exists a family of discrete valuations $(v_i)_{i\in I}$ on $K := Quot(A)$ such that $A$ equals the intersection of the valuation rings of the $v_i$ and such that for $x \neq 0$ almost all $v_i(x)$ vanish. For the general theory of valuations we refer to [Bou98], VI.3. Any Dedekind ring is a noetherian regular Krull domain.

Let $A$ be a Krull domain and $P(A)$ the set of height one prime ideals of $A$. The free abelian group $D(A)$ on $P(A)$ is called the *divisor group* of $A$. Given $P \in P(A)$ the localization $A_P$ of $A$ is a discrete valuation ring. Let $v_P$ be the associated valuation. The *divisor map*

$$div_A : K^\times \longrightarrow D(A)\ ,\ x \mapsto \sum_{P\in P(A)} v_P(x)P$$

is then a well-defined group homomorphism giving rise to the *divisor class group* $C(A) := D(A)/\mathrm{im}\ div_A$. There is a commutative diagramm of abelian groups with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A^\times \arrow[r, "\subseteq"] \arrow[d, "="] & K^\times \arrow[r, "div_A"] \arrow[d, "="] & Cart(A) \arrow[r] \arrow[d, "\nu"] & Pic(A) \arrow[r] \arrow[d, "\bar{\nu}"] & 0 \\ 0 \arrow[r] & A^\times \arrow[r, "\subseteq"] & K^\times \arrow[r, "div_A"] & D(A) \arrow[r] & C(A) \arrow[r] & 0 \end{tikzcd} \tag{2}$$

where $Cart(A)$ refers to the group of invertible fractional ideals of $A$ and $div_A : K^\times \to Cart(A)$ refers to the obvious map. Finally, $\nu : Cart(A) \to D(A), I \mapsto \sum_{P\in P(A)} v_P(I)P$ where $v_P(I)$ equals the order of the extended fractional ideal in the discrete valuation field $Quot(A_P)$. The maps $\nu$ and $\bar{\nu}$ on $Cart(A)$ and $Pic(A)$ are injective and bijective if $A$ is noetherian regular ( [Wei], Cor. I.3.8.1). If $A' \subseteq A$ is a subring which is a Krull domain itself and if $P' \in P(A')$, $P \in P(A)$ with $P \cap A' = P'$ let $e(P/P') \in \mathbb{N}$ denote the ramification index of $P$ over $P'$. Then $j(P') := \sum e(P/P')P$ induces a well-defined group homomorphism $j : D(A') \to D(A)$ where the sum runs through all $P \in P(A), P \cap A' = P'$. If $A' \to A$ is integral $j$ factors through a group homomorphism $\bar{\jmath} : C(A') \to C(A)$ ( [Bou98], Prop. VII.1.10.14).

3.2. **$p$-radical descent.** Recall that a *higher derivation* $\partial$ of *rank* $m \geq 0$ on a commutative unital ring $A$ is an ordered tuple $(\partial_0, ..., \partial_m)$ of additive endomorphisms of $A$ satisfying $\partial_0(a) = a$ together with the convolution formula

$$\partial_k(ab) = \sum_{j=0,\ldots,k} \partial_j(a)\partial_{k-j}(b) \tag{3}$$

for all $a, b \in A$ (cf. [Jac75], IV.9). Letting $A[T : m] := A[T]/(T^{m+1})$ be the truncated polynomial ring over $A$ there is an associated ring homomorphism $A \to A[T : m]$ defined by $a \mapsto \sum_{j=0,\ldots,m} \partial_j(a)T^j$ and which we will denote by $\partial$ as well. An element $a \in A$ such that $\partial(a) = a$ is called a $\partial$-*constant*. If $C \subseteq A$ is a subring with $\partial_j(C) \subseteq C$ for all $j = 0, .., m$ the ring $C$ is called *invariant*. The *order* $\mu(\partial)$ of $\partial$ is the minimal $j \geq 1$ with $\partial_j \neq 0$. If $char\, A = p > 0$ let $n(\partial) := \min\{n' : m < p^{n'}\}$.

After these preliminaries we turn to the general setting of $p$-radical descent theory of higher exponent (following [Bab81]). For the classical theory of $p$-radical descent of exponent one we refer to [KO74], II.§6. Consider a Krull domain $A$ of characteristic $p$, $K := Quot(A)$ and a higher derivation $\partial$ of rank $m$ on $K$ leaving $A$ invariant. Let $K' \subseteq K$ be the set of $\partial$-constants and let $A' := A \cap K'$. Then $A'$ is again a Krull domain, $K'$ is a field with $K' = Quot(A')$ and $A$ is finite over $A'$ (cf. [Bab81], (1.5)).

**Lemma 3.1.** *Suppose that $A$ is free over $A'$. Let $C$ be a commutative unital $A'$-algebra such that $A_C := A \otimes_{A'} C$ is an integral domain. Then $Quot(A_C)$ admits a higher derivation $\partial_C$ of the same rank and order as $\partial$ leaving $A_C$ invariant. The set of $\partial_C$-constants equals the field $Quot(C)$ and one has*

$$[Quot(A_C) : Quot(C)] = [K : K'].$$

*Proof:* Let $S := C - \{0\}$. Each base extension $\partial_j \otimes id : A_C \to A_C$ composed with the natural map $A_C \to S^{-1}A_C$ extends to an additive endomorphism of $S^{-1}A_C$. By construction the convolution formula (3) holds whence these endomorphisms form a higher derivation on $S^{-1}A_C$ of the same rank and order as $\partial$ leaving $A_C$ invariant. Since $A_C$ is an integral domain and integral over $C$ one has $S^{-1}A_C = Quot(A_C)$ (cf. [Mat86], Lem. 3.9.1). The field of $\partial_C$-constants equals $(S^{-1}(A_C))^{\partial_C} = S^{-1}(A^{\partial} \otimes_{A'} C) = Quot(C)$ where the first identity holds by freeness of $A$ over $A'$. $\square$

We will associate to the couple $(A, \partial)$ the following construction. As in the previous subsection the integral extension $A' \to A$ gives rise to the canonical homomorphisms $j : D(A') \to D(A)$ and $\bar{\jmath} : C(A') \to C(A)$. On the other hand there is the abelian group of "logarithmic derivatives" $\mathcal{L} := \{\partial(z)/z, z \in K^\times\} \subseteq K[T : m]^\times$ and we may form the subgroups

$$\begin{aligned} \mathcal{L}_A &:= \mathcal{L} \cap A[T : m]^\times, \\ \mathcal{L}'_A &:= \{\partial(u)/u, u \in A^\times\} \end{aligned} \tag{4}$$

with $\mathcal{L}'_A \subseteq \mathcal{L}_A$. There is an injective (cf. [Bab81], (1.5)) homomorphism

$$\Phi_A : \ker \bar{\jmath} \longrightarrow \mathcal{L}_A/\mathcal{L}'_A \ , \ D \bmod \ div_{A'}(K'^\times) \mapsto \partial(x_D)/x_D \bmod \mathcal{L}'_A$$

where as before $x_D \in K^\times$ is chosen such that $div_A(x_D) = j(D)$. Consider the following condition on the tuple $(A, \partial)$

$$\text{(HYP)} \qquad [K : K'] = p^{n(\partial)} \text{ and } \partial_{\mu(\partial)}(a) \in A^\times \text{ for some } a \in A.$$

If (HYP) holds the main result of [loc.cit.], Thm. 1.6 yields that $\Phi_A$ is surjective.

**Lemma 3.2.** *Suppose $A$ is free over $A'$ and let $C$ be a commutative unital $A'$-algebra such that $A_C$ is again a Krull domain. Let $(A_C, \partial_C)$ be as in Lem. 3.1. If $(A, \partial)$ satisfies* (HYP) *then $(A_C, \partial_C)$ satisfies (HYP) whence $\Phi_{A_C}$ is bijective.*

*Proof:* This is clear from the construction and Lem. 3.1. $\square$

For future reference we emphasize the following special case. Consider the extension of Krull domains of characteristic $p$

$$A' := k[t^{p^m}, t^{-p^m}] \subseteq k[t, t^{-1}] =: A \tag{5}$$

where $k$ is a finite field and $m \geq 0$. Put $K' := Quot(A')$ and $K := Quot(A) = K'[t]$. For $j \in \mathbb{N}_0$ let $\partial_j$ be the $K'$-linear endomorphism on $K$ whose effect on the $K'$-basis $1, t, ..., t^{m'}$ (with $m' := p^m - 1$) is given by

$$\partial_j t^i = \binom{i}{j} t^{i-j}$$

where $\binom{i}{j} = 0$ if $j > i$. This is the standard higher derivation on $K$ (cf. [Jac75], IV.9).

**Lemma 3.3.** *The higher derivation $\partial := (\partial_0, ..., \partial_{m'})$ has rank $m'$, $\mu(\partial) = 1$, $n(\partial) = m$ and leaves $A$ invariant. The field of constants equals $K'$ and $K' \cap A = A'$. The couple $(A, \partial)$ satisfies* (HYP).

*Proof:* The statements about rank, order and $n(\partial)$ are immediate. We show that $K'$ indeed equals the field of constants $K''$. The extension $K/K''$ is purely inseparable since $K' \subseteq K''$. Hence the minimal polynomial of $t$ over $K''$ has the form $X^{p^l} - a,\ a \in K'',\ l \geq 0$ and it suffices to see that $l = m$. Suppose $l < m$. Then $t^{p^l} \in K''$ but $\partial_{p^l}(t^{p^l}) = 1 \neq 0$, a contradiction. The claim $K' \cap A = A'$ is immediate. Finally, $[K : K'] = rk_{A'} A = p^m = p^{n(\partial)}$ and $\partial_1(t) = 1$. Thus the couple $(A, \partial)$ satisfies (HYP). $\square$

## 4. Picard groups of twisted affinoid discs

We use $p$-radical descent theory to obtain a structure result on the Picard group of certain twisted affinoid discs.

Following ideas of Temkin in [Tem04], §3 we begin by introducing some graded notions in the setting of affinoid algebras. Let $K$ be a discretely valued complete nonarchimedean field of characteristic zero with residue field $k$ and with algebraic closure $\bar{K}$. Let $A$ be a $K$-affinoid algebra. The submultiplicative spectral seminorm $|.|$ induces a decreasing complete and exhaustive filtration on $A$ by the additive subgroups

$$F^s A := \{a \in A,\ |a| \leq p^{-s}\}.$$

Thus, $A$ becomes a filtered ring in the sense of section 2 and Noether normalization shows the filtration to be quasi-integral. Let $gr^\bullet A := \oplus_{s \in \mathbb{R}} gr^s A$ be the graded ring where as before $gr^s A := F^s A / F^{s+} A$. In particular $gr^0 A$ equals the usual reduction (cf. [BGR84], 6.3) of $A$. Clearly $gr^\bullet A$ is a graded $gr^\bullet K$-algebra and $gr^\bullet$ is functorial with respect to morphisms between $K$-affinoid algebras (since such morphisms are automatically spectral seminorm decreasing, cf. [loc.cit.], Prop. 6.2.2/1). For a nonzero $a \in A$ denote by $\deg(a) \in \mathbb{R}$ the *degree* of $a$ in the filtration. The *principal symbol* $\sigma(a) \neq 0$ of $a$ is then given by $a + F^{s+} A \in gr^s A \subseteq gr^\bullet A$ where $s = \deg(a)$. For example

$$gr^\bullet K \xrightarrow{\cong} k[t, t^{-1}],\ \sigma(\pi) \mapsto t \tag{6}$$

where $\pi$ is a prime element in $K$ and $t$ a variable. If in this situation $K \subseteq K_1$ is a finite extension then the natural homomorphism of $gr^\bullet K_1$-algebras

$$gr^\bullet A \otimes_{gr^\bullet K} gr^\bullet K_1 \longrightarrow gr^\bullet (A \otimes_K K_1) \tag{7}$$

induced by the isometry $A \to A \otimes_K K_1, a \mapsto a \otimes 1$ is bijective. Indeed, being discretely valued $K$ is stable whence the extension $K \subseteq K_1$ is $K$-cartesian (in the sense of [BGR84], Def. 2.4.1/1) and the claim follows from [LvO96], Lem. I.6.14. By [loc.cit.], Cor. I.7.2.2 we have

**Lemma 4.1.** *Let $A$ be reduced and let the ring $gr^\bullet A$ be noetherian and regular. Then $A$ is regular with global dimension* $\operatorname{gld} A \leq \operatorname{gld} gr^\bullet A$.

For $r \in |\bar{K}^\times|$ denote by $\mathbf{B}(r)$ the $K$-affinoid space equal to the closed disc of radius $r$ around zero. For our purposes, a $K$-affinoid space $X$ is called a *twisted form* of $\mathbf{B}(r)$ if there is a finite Galois extension $K \subseteq K_1 \subseteq \bar{K}$ and an isomorphism $X \otimes_K K_1 \cong \mathbf{B}(r) \otimes_K K_1$ of $K_1$-affinoids (cf. [Ser02], III. §1). In this situation we say that the form $X$ is *trivialized* by the extension $K \subseteq K_1$.

After these preliminaries we fix a form $X$ of $\mathbf{B}(r)$ and a trivializing extension $K_1$. Denote by $A := \mathcal{O}(X)$ and $B := \mathcal{O}(\mathbf{B}(r) \otimes_K K_1)$ the affinoid algebras of $X$ and $\mathbf{B}(r) \otimes_K K_1$ and by $gr^\bullet A$ and $gr^\bullet B$ the corresponding Temkin reductions.

**Lemma 4.2.** *The ring $A$ is a Dedekind domain. The ring $gr^\bullet A$ is a noetherian regular integral domain of global dimension* 2. *There is a canonical isomorphism $Pic(A) \xrightarrow{\cong} Pic(gr^\bullet A)$.*

*Proof:* The injective ring homomorphism $A \to B$ is finite free and $B$ is Dedekind. Standard commutative algebra (cf. [Mat86], 3.9) yields that $A$ is Dedekind. Using (7) the homomorphism of $gr^\bullet K$-algebras

$$gr^\bullet A \otimes_{gr^\bullet K} gr^\bullet K_1 \longrightarrow gr^\bullet (A \otimes_K K_1) \cong gr^\bullet B$$

is bijective. Let $z \in B$ be a parameter on the disc. The ring $gr^\bullet B = (gr^\bullet K_1)[\sigma(z)]$ is a noetherian regular integral domain of global dimension 2 according to (6). At the same time it is a finite free module over $gr^\bullet A$ whence the ring $gr^\bullet A$ itself enjoys the same properties (cf. [Bou98], Cor. I.3.5, [MR87], Thm. 7.2.8). Thus, $A$ is a one dimensional integral Zariski ring with regular integral graded ring and Lem. 2.3 gives the desired isomorphism $Pic(A) \xrightarrow{\cong} Pic(gr^\bullet A)$. $\square$

Let $k$ and $k_1$ be the residue fields of $K$ and $K_1$ respectively and denote by $k(t_0)$ the function field in one variable $t_0$ over $k$. Let $\pi$ be a uniformizer for $K_1$ so that we obtain by (6) an extension

$$gr^\bullet K \cong k[t_0^{\pm e'}] \xrightarrow{\subseteq} k_1[t_0^{\pm 1}] \cong gr^\bullet K_1 \tag{8}$$

where $e' = e(K_1/K)$ denotes the ramification index. Choose $m \geq 0$ such that $p^m e_0' = e'$ and $(p, e_0') = 1$. By (7) the space $Spec(gr^\bullet A)$ is a form of the affine line $Spec(gr^\bullet B)$ with respect to the extension (8). Since there are no separable forms of the affine line over a field (cf. [Rus70], 1.1) the cartesian diagram

$$\begin{array}{ccc} k[t_0^{\pm e_0'}] & \xrightarrow{\subseteq} & k_1[t_0^{\pm 1}] \\ {\scriptstyle\subseteq}\downarrow & & {\scriptstyle\subseteq}\downarrow \\ k(t_0^{e_0'}) & \xrightarrow{\subseteq} & k_1(t_0) \end{array}$$

implies that $Spec(gr^\bullet A)$ is trivialized by the inseparable closure $(gr^\bullet K)^{p^{-m}} = k[t^{\pm 1}]$ inside (8) where $t := t_0^{e_0'}$. The trivializing extension is therefore of the form (5) and comes equipped with the standard higher derivation $\partial$ of rank $m' := p^m - 1$

satisfying (HYP) (cf. Lem. 3.3). Let $z \in B$ be a parameter and denote its principal symbol $z \in gr^\bullet B$ by $z$ as well. We may base extend $\partial$ to $gr^\bullet A \otimes_{gr^\bullet K} (gr^\bullet K)^{p^{-m}} \simeq k[t^{\pm 1}, z]$ to obtain a higher derivation $\partial'$ on $k(t,z)$ of the same rank, leaving $C := k[t^{\pm 1}, z]$ invariant and satisfying (HYP) (cf. Lemmas 3.1/3.2). Thus

**Proposition 4.3.** *Denoting by $C(gr^\bullet A)$ the divisor class group of $gr^\bullet A$ the homomorphism $\Psi_C : C(gr^\bullet A) \xrightarrow{\cong} \mathcal{L}_C/\mathcal{L}'_C$ associated to $\partial'$ is bijective.*

To calculate the right-hand side of the above bijection we need information on the special value $\partial'(z) \in k[t^{\pm 1}, z]$. This requires an auxiliary lemma. Let $n := [K_1 : K]$. Denoting by $K_1^*$ the dual $K$-vector space of $K_1$ we have $K_1 \otimes_K K_1^* \xrightarrow{\cong} \mathrm{End}_K(K_1)$ in the usual way. On the other hand, $K \subseteq K_1$ being Galois implies that the $K_1$-module $\mathrm{End}_K(K_1)$ is finite free on the basis $G := Gal(K_1/K)$ (cf. [KO74], §5). For any $K$-basis $\{a_1, ..., a_n\}$ of $K_1$ denote by $\{a_1^*, ..., a_n^*\}$ the dual basis.

**Lemma 4.4.** *Let $v := (c(\sigma))_{\sigma \in G}$ be any element in $K_1^n - \{0\}$. There exists an orthogonal $K$-basis (depending on $v$) of $K_1$ such that with $1 \otimes a_i^* =: \sum_\sigma s_\sigma^i \sigma$, $s_\sigma^i \in K_1$ we have $\sum_\sigma s_\sigma^i c(\sigma) \neq 0$ in $K_1$ for $i = 1, ..., n$.*

*Proof:* Let $\underline{a} := (a_1, ..., a_n) \in K_1^n$ have entries a $K$-basis of $K_1$ (orthogonal or not) and denote by $G(\underline{a})$ the matrix $(\sigma a_j)_{\sigma, j}$. By independance of characters we have $G(\underline{a}) \in \mathrm{GL}_n(K_1)$. By definition of the $s_\sigma^i$ we have with $s^i := (s_\sigma^i)_\sigma \in K_1^n$ that $G(\underline{a})^t s^i = e_i$, the $i$-th standard unit vector in $K_1^n$. It follows that $G(\underline{a})^{-1} = (s_\sigma^i)_{i,\sigma}$. Let $H_i$ be the hyperplane $e_i = 0$ in $K_1^n$ and let $H := \cup_i H_i$. The claim amounts to find an orthogonal $K$-basis $a_1, ..., a_n$ of $K_1$ such that $w(\underline{a}) := G(\underline{a})^{-1} v \notin H$. If $A \in \mathrm{GL}_n(K)$ then a short calculation yields

$$(A^t)^{-1} G(\underline{a})^{-1} = G(A\underline{a})^{-1}$$

where $A\underline{a} \in K_1^n$ has again entries a $K$-basis of $K_1$. Now $K_1$ being $K$-cartesian we may choose an orthogonal $K$-basis $a_1, ..., a_n$ of $K_1$. By the above it suffices to find an element $A \in \mathrm{GL}_n(K)$ such that $(A^t)^{-1} w(\underline{a}) \notin H$ and such that $A\underline{a} \in K_1^n$ has entries an orthogonal basis. Now the principal symbols $\tilde{a}_i$ form a $gr^\bullet K$-basis of $gr^\bullet K_1$. For all $i, j = 1, ..., n$ let $a_{ij}$ be a principal unit of $K$ if $i = j$ and an arbitrary element of $K$ such that $|a_{ij} a_j| < |a_{ii} a_i|$ if $i \neq j$ respectively. Let $A := (a_{ij}) \in Mat(n, K)$. Since the principal symbol of $\sum_j a_{ij} a_j$ equals $\tilde{a}_{ii} \tilde{a}_i$ for all $i$ the entries of $A\underline{a} \in K_1^n$ are an orthogonal $K$-basis of $K_1$. Choosing the non-diagonal entries sufficiently small $A$ becomes invertible and we have $(A^t)^{-1} = 1 + M$ with $M \in Mat(n, K)$ close to zero. Conversely, we see that any such $M$ gives rise to a matrix $A \in \mathrm{GL}_n(K)$ such that the entries of $A\underline{a} \in K_1^n$ are an orthogonal $K$-basis of $K_1$. Since $w(\underline{a}) \neq 0$ we clearly have $w(\underline{a}) + M w(\underline{a}) \notin H$ for some choice of $M$. □

**Lemma 4.5.** *For $j = 0, ..., m'$ there exists $c_j \in k[t^{\pm 1}]$ such that $\partial'_j(z) = c_j z$.*

*Proof:* To ease notation we put $\tilde{R} := gr^\bullet R$ for any occuring filtered ring $R$. Consider the trivializing extension $K \subseteq K_1$ for our fixed form $A$ of $B$. Consider the composite of finite free ring extensions

$$\tilde{K} = k[t^{\pm p^m}] \longrightarrow (\tilde{K})^{p^{-m}} = k[t^{\pm 1}] \longrightarrow \tilde{K}_1 = k_1[t_0^{\pm 1}]$$

where as above $k$ and $k_1$ denote the residue fields of $K$ and $K_1$ respectively, $t_0 \in \tilde{K}_1$ is the principal symbol of a uniformizer of $K_1$ and $t := t_0^{e'_0}$. Let

$$f \in \mathrm{Hom}_{K_1 \otimes_K K_1}(K_1 \otimes_K B, B \otimes_K K_1)$$

be the faithfully flat descent datum for the form $A \otimes_K K_1 \simeq B$. Since $K_1$ is $K$-cartesian we have $gr^\bullet (K_1 \otimes_K B) = \tilde{K}_1 \otimes_{\tilde{K}} \tilde{B}$ for the tensor filtration making it a

$gr^\bullet(K_1 \otimes_K K_1) = \tilde{K}_1 \otimes_{\tilde{K}} \tilde{K}_1$-algebra (cf. [LvO96], Lem. I.6.14). By construction (cf. [KO74], §8) $f$ is a filtered homomorphism and

$$\tilde{f} := gr^\bullet f \in \mathrm{Hom}_{\tilde{K}_1 \otimes_{\tilde{K}} \tilde{K}_1}(\tilde{K}_1 \otimes_{\tilde{K}} \tilde{B}, \tilde{B} \otimes_{\tilde{K}} \tilde{K}_1)$$

yields the faithfully flat descent datum for the form $\tilde{A} \otimes_{\tilde{K}} \tilde{K}_1 \simeq \tilde{B}$. Denote by $\tilde{z} \in \tilde{B}$ the principal symbol of the above chosen parameter $z \in B$. Let $l \in \mathbb{N}$ be any natural number. In the following we obtain information on the element $\tilde{f}(1 \otimes \tilde{z}^l) \in \tilde{B} \otimes_{\tilde{K}} \tilde{K}_1$. The Galois descent datum $\bar{\sigma} : B \to B,\ \sigma \in G := Gal(K_1/K)$ gives a $K_1 \otimes_K K_1$-linear endomorphism

$$T : \mathrm{End}_K(K_1) \longrightarrow \mathrm{End}_K(B)$$

via $T(s\sigma)(b) = s\bar{\sigma}(b)$ for $s \in K_1,\ \sigma \in G$ (a descent datum "fidèlement projective", cf. [KO74], Thm. 5.1). Applying [loc.cit.], Prop. 4.1 we see that

$$f(1 \otimes z^l) = \sum_{i=1}^{n} T(1 \otimes a_i^*)(z^l) \otimes a_i \in B \otimes_K K_1 \tag{9}$$

where $a_1, ..., a_n$ is any $K$-basis of $K_1$ and $a_1^*, ..., a_n^*$ the dual basis. Furthermore, let $\theta : G \to \mathrm{Aut}_{K_1-alg}(B)$ be the Galois cocycle corresponding to the descent datum $\{\bar{\sigma}\}_{\sigma \in G}$ and let

$$\theta_\sigma(Z) = c_0(\sigma) + c_1(\sigma)Z + ... \in K_1[[Z]]$$

be the power series giving the corresponding algebra automorphism of $B$. By [BGR84], Cor. 5.1.4/10 we have $|c_0(\sigma)| \leq 1, |c_1(\sigma)z| = 1, |c_i(\sigma)z^i| < 1$ for $i > 1$ where $|.|$ denotes the spectral norm on $B$. We let $v := (c_1(\sigma))_{\sigma \in G} \in K_1^n - \{0\}$ and let from now on $a_1, ..., a_n$ be an orthogonal $K$-basis of $K_1$ which is adapted to $v$ in the sense of the preceding lemma. With

$$1 \otimes a_i^* =: \sum_\sigma s_\sigma^i \sigma \in \mathrm{End}_K(K_1) \tag{10}$$

this implies $\sum_\sigma s_\sigma^i c_1(\sigma) \neq 0$ for $i = 1, ..., n$. Since $\tilde{f}(1 \otimes \tilde{z}^l) \neq 0$ it must equal the principal symbol $f(1 \otimes z^l)^\sim$ of $f(1 \otimes z^l)$ whence by (9)

$$\tilde{f}(1 \otimes \tilde{z}^l) = \sum_{i \in S_l} T(1 \otimes a_i^*)(z^l)^\sim \otimes \tilde{a}_i$$

where $\sum_{i \in S_l}$ refers to the sum over the terms of dominant degree in the filtration. Using (10) and $\theta_\sigma \circ \sigma = \bar{\sigma}$ as maps $B \to B$ for $\sigma \in G$ the definition of $T$ yields

$$T(1 \otimes a_i^*)(z^l)^\sim = (\sum_{\sigma \in G} T(s_\sigma^i \sigma)(z^l))^\sim = (\sum_{\sigma \in G} s_\sigma^i \theta_\sigma(z^l))^\sim$$

with $\sum_{\sigma \in G} s_\sigma^i \theta_\sigma(z) =: p_i + d_i Z + ... \in K_1[[Z]]$ and $d_i := \sum_\sigma s_\sigma^i c_1(\sigma) \neq 0$ for all $i = 1, ..., n$. We assume first $p_i = 0$ for all $i = 1, ..., n$. Since the spectral norm on $B$ is given via $|\sum_m h_m z^m| = \sup_m |h_m| r^m$ with some $r < 1$ ( [BGR84], Prop. 6.1.5/5) there is $N \geq 0$ such that for all $l \geq N$ we have $(\sum_\sigma s_\sigma^i \theta_\sigma(z^l))^\sim = \tilde{d}_i \tilde{z}^l$ for all $i = 1, ..., n$. All in all we obtain

$$\tilde{f}(1 \otimes \tilde{z}^l) = \sum_{i \in S_l} \tilde{d}_i \tilde{z}^l \otimes \tilde{a}_i \tag{11}$$

for $l \geq N$. We now use this information to prove the lemma.

Since $\tilde{K} \to \tilde{K}_1$ is finite free $\tilde{f}$ uniquely determines a $\tilde{K}_1 \otimes_{\tilde{K}} \tilde{K}_1$-linear homomorphism (cf. [KO74], Thm. 4.3)

$$\tilde{T} : \mathrm{End}_{\tilde{K}}(\tilde{K}_1) \longrightarrow \mathrm{End}_{\tilde{K}}(\tilde{B})$$

and (11) implies

$$\tilde{T}(s\otimes s^*)(\tilde{z}^l)=\sum_{i\in S_l} s\,s^*(\tilde{a}_i)\tilde{d}_i\tilde{z}^l=(\sum_{i\in S_l} s\,s^*(\tilde{a}_i)\tilde{d}_i)\tilde{z}^l\in\tilde{B}$$

for $l\geq N$. Recall the standard higher derivation $\partial$ on $k(t)$ leaving $k[t^{\pm 1}]$ invariant. Since $(\tilde{K})^{p^{-m}}$ is a direct summand of the $\tilde{K}$-module $\tilde{K}_1$ we may extend for $j=0,...,m'$ the $\tilde{K}$-linear endomorphism $\partial_j$ of $k[t^{\pm}]$ to an element of $\mathrm{End}_{\tilde{K}}(\tilde{K}_1)$. In the following fix $j\in\{0,...,m'\}$ and write $\partial_j=:\sum_{s,s^*}s\otimes s^*\in\tilde{K}_1\otimes_{\tilde{K}}\tilde{K}_1=\mathrm{End}_{\tilde{K}}(\tilde{K}_1)$. By construction of $\partial'_j$ we have $\partial'_j=\tilde{T}(\partial_j)\in\mathrm{End}_{\tilde{K}}(\tilde{B})$ (cf. [KO74], Thm. 4.3) whence for $l\geq N$

$$\partial'_j(\tilde{z}^l)=\sum_{s,s*}\tilde{T}(s\otimes s^*)(\tilde{z}^l)=(\sum_{s,s^*}\sum_{i\in S_l}s\,s^*(\tilde{a}_i)\tilde{d}_i)\tilde{z}^l=:c_j(l)\tilde{z}^l\in\tilde{B} \tag{12}$$

with $c_j(l)\in(\tilde{K})^{p^{-m}}$ (the latter since $\partial'_j$ leaves $k[t^{\pm 1},\tilde{z}]$ invariant). Since $\tilde{K}\subseteq(\tilde{K})^{p^{-m}}$ is radical of exponent $p^m$ the endomorphism $\partial'_j$ of $k[t^{\pm 1},\tilde{z}]$ is $k[t^{\pm},\tilde{z}]^{p^m}$-linear. For $lp^m+1\geq N$ we obtain

$$\tilde{z}^{lp^m}\partial'_j(\tilde{z})=\partial'_j(\tilde{z}^{lp^m+1})=c_j(lp^m+1)\tilde{z}^{lp^m+1} \tag{13}$$

in the integral domain $k[t^{\pm 1},\tilde{z}]$ whence $\partial'_j(\tilde{z})=c_j\tilde{z}$ and $c_j:=c_j(lp^m+1)$ does not depend on $l$. The lemma is thus proved under the hypothesis that $p_i=0$ for all $i=1,...,n$. In general, we see that the presence of some $p_i\neq 0$ leads to a modification in line (12) in the sense that $\partial'_j(\tilde{z}^l)=e_j(l)+c_j(l)\tilde{z}^l\in\tilde{B}$ with some additional $e_j(l)\in(\tilde{K})^{p^{-m}}$. For $lp^m+1\geq N$ we obtain again

$$\tilde{z}^{lp^m}\partial'_j(\tilde{z})=\partial'_j(\tilde{z}^{lp^m+1})=e_j(lp^m+1)+c_j(lp^m+1)\tilde{z}^{lp^m+1}$$

whence $e_j(lp^m+1)=0$. The general case follows. $\square$

By the preceding proposition we have $\partial'(z)=\sum_{j=0}^{m'}c_jzT^j$ in $k[t^{\pm 1},z][T:m']^\times$ and therefore $\partial'(z)/z\in\mathcal{L}_C$.

**Proposition 4.6.** *The class of $\partial'(z)/z$ generates the group $\mathcal{L}_C/\mathcal{L}'_C$. Hence, $C(gr^\bullet A)$ is finite cyclic of order dividing $p^m$.*

*Proof:* The second claim follows from the first by Prop. 4.3 since $\mathcal{L}_C$ has exponent $m'+1=p^m$. Given $f,g\in k[t^{\pm 1},z]-\{0\}$ we have $\partial'(f/g)/(f/g)=\partial'(fg^{m'})/fg^{m'}$ whence

$$\mathcal{L}_C=\{\partial'(f)/f\mid f\in k[t^{\pm 1},z]-\{0\},\ \partial'(f)/f\in(k[t^{\pm 1},z][T:m'])^\times\}.$$

Since $\partial'$ is a ring homomorphism we have $\mathrm{ord}_z(f)\leq\mathrm{ord}_z(\partial'(f))$ for $f\in k[t^{\pm 1},z]-\{0\}$. Thus, $\partial'(f)/f\in k[t^{\pm 1},z][T:m']^\times$ implies $\partial'(f)/f\in k[t^{\pm 1}][T:m']^\times$. Now let $f=\sum_{i,j}a_{ij}t^iz^j\in k[t^{\pm 1},z]-\{0\}$ with $a_{ij}\in k^\times$ and assume $\partial'(f)/f\in\mathcal{L}_C$. Hence $\partial'(f)/f=:h\in k[t^{\pm 1}][T:m']^\times$. Letting $w:=\sum_{j=0}^{m'}c_jT^j$ we compute

$$\sum_{i,j}a_{ij}(t+T)^iz^jw^j=\sum_{i,j}a_{ij}\partial'(t)^i\partial'(z)^j=\partial'(\sum_{i,j}a_{ij}t^iz^j)=h\sum_{i,j}a_{ij}t^iz^j.$$

Fix a $j$. Since $t,z,T$ are algebraically independent one has

$$w^j\,(\sum_i a_{ij}(t+T)^i)/(\sum_i a_{ij}t^i)=h.$$

Since $w=\partial'(z)/z\in\mathcal{L}_C$ we have

$$\sum_i a_{ij}(t+T)^i)/(\sum_i a_{ij}t^i)=\partial(\sum_i a_{ij}t^i)/(\sum_i a_{ij}t^i)\in\mathcal{L}_{k[t^{\pm 1}]}$$

where $\mathcal{L}_{k[t^{\pm 1}]}$ refers to the integral logarithmic derivatives associated to the standard derivation $\partial$. Since the latter satisfies (HYP) and $C(k[t^{\pm 1}]) = 0$ we have $\mathcal{L}_{k[t^{\pm 1}]} = \mathcal{L}'_{k[t^{\pm 1}]} \subseteq \mathcal{L}'_C$. Thus, $w^j \equiv\ h \bmod \mathcal{L}'_C$ in $\mathcal{L}_C/\mathcal{L}'_C$. □

Summing up we have

**Theorem 4.7.** *The group $Pic(A)$ is finite cyclic of order dividing $p^m$.*

The preceding result also determines the Grothendieck group $K_0(A)$. Indeed, $A$ is Dedekind whence $SK_0(A) = 0$ by Lemma 2.2.

## 5. Applications to $p$-adic Fourier theory

5.1. **Cohomology and inverse limits.** We establish some results on the behaviour of sheaf cohomology with respect to certain coverings.

Let $X$ be a $G$-ringed space in the sense of [BGR84], 9.3.1 and let $H^*(X,.)$ denote the sheaf cohomology of $X$. Let $Ab(X)$ be the category of abelian sheaves on $X$, $Ab$ the category of abelian groups, $Proj_{\mathbb{N}}(Ab)$ the category of $\mathbb{N}$-projective systems over $Ab$ and $F := \varprojlim_n$ the projective limit viewed as an additive functor $Proj_{\mathbb{N}}(Ab) \to Ab$.

**Lemma 5.1.** *Let $\{X_n\}_{n\in\mathbb{N}}$ be a countable increasing admissible open covering of $X$. Consider the additive functor*

$$G : Ab(X) \to Proj_{\mathbb{N}}(Ab),\ \mathcal{F} \mapsto (\mathcal{F}(X_n))_n.$$

*For any $\mathcal{F} \in Ab(X)$ there is a short exact sequence*

$$0 \longrightarrow (R^1F)(G\mathcal{F}) \longrightarrow R^1(FG)(\mathcal{F}) \longrightarrow (FR^1G)(\mathcal{F}) \longrightarrow 0.$$

*Proof:* Both $F$ and $G$ are left exact. An injective sheaf $\mathcal{I} \in Ab(X)$ is flasque (cf. [Har77], Lem. II.2.4) whence the system $G(\mathcal{I})$ is surjective and therefore $F$-acyclic (cf. [Wei94], Lem. 3.5.3/Cor. 3.5.4). The standard Grothendieck spectral sequence (cf. [loc.cit.], Thm. 5.8.3)

$$E_2^{pq} := (R^pF)(R^qG)(\mathcal{F}) \Rightarrow R^{p+q}(FG)(\mathcal{F})$$

is therefore at our disposal. The associated five-term exact sequence degenerates into our short exact sequence since $R^2F = 0$ (cf. [loc.cit.], Cor. 3.5.4). □

**Lemma 5.2.** *Keep the assumptions of Lem. 5.1. Let $res_n : Ab(X) \to Ab(X_n)$ be the restriction. For $i \geq 0$ the collection of functors*

$$T^i := (H^i(X_n, res_n \circ (.)))_n$$

*form a universal cohomological $\delta$-functor $(T^i)_{i\geq 0} : Ab(X) \to Proj_{\mathbb{N}}(Ab)$. For any $\mathcal{F} \in Ab(X)$ there is a short exact sequence*

$$0 \longrightarrow \varprojlim_n^{(1)} \mathcal{F}(X_n) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow \varprojlim_n H^1(X_n, \mathcal{F}) \longrightarrow 0$$

*where the right surjection is induced by the inclusions $X_n \to X$ for each $n \in \mathbb{N}$.*

*Proof:* For each fixed $n \in \mathbb{N}$ the functors $H^i(X_n, res_n \circ (.)), i \geq 0$ form a cohomological $\delta$-functor $Ab(X) \to Ab$. Since it is functorial in $X_n$ it follows from the definition of short exact sequences in $Proj_{\mathbb{N}}(Ab)$ that the $T^i = (H^i(X_n, res_n \circ (.)))_n$ for $i \geq 0$ form a cohomological $\delta$-functor $(T^i)_{i\geq 0} : Ab(X) \to Proj_{\mathbb{N}}(Ab)$. Now on any $G$-ringed space the restriction of an injective sheaf to an admissible open subset remains injective (cf. [Har75], I.§2). Hence each $res_n$ preserves injectives. Since $Ab(X)$ has enough injectives it follows that each $T^i,\ i > 0$ is effaceable and therefore $(T^i)_{i\geq 0}$ is universal (cf. [Gro57], II.2.2.1). It now follows that the right-derived functors of the functor $G$ appearing in Lem. 5.1 are given by $(T^i)_{i\geq 0}$. It remains

to observe that the sheaf property yields $F \circ G = \Gamma$, the global section functor. $\square$

We apply the above results to certain rigid analytic spaces. Let $K$ be a completely valued non-archimedean field of characteristic zero and $Rig(K)$ be the category of rigid $K$-analytic spaces (cf. [BGR84], 9.3). As with any $G$-ringed space, given $X \in Rig(K)$ its *Picard group* $Pic(X)$ equals the set of isomorphism classes of line bundles over $X$ with the tensor product as binary operation. Similarly the *Grothendieck group* $K_0(X)$ of $X$ equals the group completion of the abelian mononoid consisting of isomorphism classes of vector bundles over $X$ with the direct sum as binary operation (cf. [FvdP04], 4.7). Analogously to the algebraic setting (section 2) rank and determinant of vector bundles induces a surjective group homomorphism

$$\mathrm{rk} \oplus \det : K_0(X) \longrightarrow H^0(X, \mathbb{Z}) \oplus Pic(X)$$

(cf. [Wei], II.§8). Recall also that $X$ is called *quasi-Stein* if there is a countable increasing admissible open affinoid covering $\{X_n\}_{n \in \mathbb{N}}$ of $X$ such that the restriction maps $\mathcal{O}(X_{n+1}) \longrightarrow \mathcal{O}(X_n)$ have dense image (cf. [Kie67], Def. 2.3).

**Corollary 5.3.** *Let $X$ be a quasi-Stein rigid $K$-analytic space with respect to a covering $\{X_n\}_{n \in \mathbb{N}}$. There is an exact sequence*

$$0 \longrightarrow \varprojlim_n^{(1)} \mathcal{O}(X_n)^\times \longrightarrow Pic(X) \longrightarrow \varprojlim_n Pic(X_n) \longrightarrow 0.$$

*Proof:* Apply the last result to $\mathcal{F} = \mathbb{G}_m$ and use $H^1(X, \mathbb{G}_m) = Pic(X)$ (cf. [FvdP04], Prop. 4.7.2). $\square$

For future reference we point out the relation between the $K_0$-theory of a quasi-Stein space $X \in Rig(K)$ and of its ring of global sections $\mathcal{O}(X)$. To do this recall that the *dimension* of $X$ is defined to be the supremum over the Krull dimensions of its various local rings (cf. [loc.cit.], Def. 4.5.6).

**Lemma 5.4.** *Let $X$ be a quasi-Stein rigid $K$-analytic space of finite equidimension such that $\pi_0(X)$ is finite. Taking global sections furnishes an equivalence of exact categories between vector bundles over $X$ and finitely generated projective $\mathcal{O}(X)$-modules. In particular, $K_0(X) \cong K_0(\mathcal{O}(X))$ and $Pic(X) \cong Pic(\mathcal{O}(X))$ canonically.*

*Proof:* Following [Gru68], Remarque V.1. one may apply *mutatis mutandis* the arguments appearing in the proof of [loc.cit.], Thm. V.1 on each connected component. $\square$

5.2. **The additive group.** Let $|.|$ be the absolute value on $\mathbb{C}_p$ normalized via $|p| = p^{-1}$. Let $\mathbb{Q}_p \subseteq L$ be a finite field extension (the "base field"). Let $o \subseteq L$ be the integers in $L$ and let $e_1, ..., e_n$ be a $\mathbb{Z}_p$-basis of $o$. Let $e = e(L/\mathbb{Q}_p)$ be the ramification index, $k$ be the residue field of $L$ and $q = \#k$ its cardinality. To ease notation let us put $X_F := X \hat{\otimes}_L F$ for any $X \in Rig(L)$ and any completely valued intermediate field $L \subseteq F \subseteq \mathbb{C}_p$. Let $\mathcal{G}_L$ denote the subcategory of $Rig(L)$ consisting of group objects. Denote by $\mathbf{B}^n \in Rig(L)$ the $n$-dimensional open unit disc around 0. Let $\hat{o} \subseteq \mathbf{B}^n$ denote the closed analytic subvariety defined by the equations

$$e_i \log(1 + z_j) - e_j \log(1 + z_i) = 0$$

for $i, j = 1, ..., n$. It is a connected smooth one dimensional quasi-Stein space. For each completely valued intermediate field $L \subseteq F \subseteq \mathbb{C}_p$ its $F$-valued points are in natural bijection with the set of $F$-valued locally $L$-analytic characters of the additive group of $o$ viewed as a locally $L$-analytic group. In this way $\hat{o}$ becomes naturally an object of $\mathcal{G}_L$ (for all further details we refer to [ST], §2). Let $L \subseteq$

$K \subseteq \mathbb{C}_p$ be a discretely valued complete intermediate field (the "coefficient field"). The principal goal of this section is to use the methods of the preceding section to calculate the group $Pic(\hat{o}_K)$.

Recall that a *Lubin-Tate formal group* for a fixed prime element $\pi \in o$ is a certain one dimensional commutative formal group $\mathbb{G}$ over $o$ of height $n = [L : \mathbb{Q}_p]$ (cf. [Lan90], 8.§1). It comes equipped with a unital ring homomorpism $[.] : o \to \mathrm{End}(\mathbb{G})$ where we assume ([loc.cit.], Thm. 1.1/3.1) that $[\pi] = \pi X + X^q \in o[[X]]$. Denote by $\exp_{\mathbb{G}}$ and $\log_{\mathbb{G}}$ the formal exponential and logarithm of $\mathbb{G}$ respectively. Denote by $L \subseteq L_\infty$ the extension obtained by adjoining all torsion points of $\mathbb{G}$ and all $p$-th roots of unity to $L$. Viewing $\mathbb{G}$ as a connected $p$-divisible group (cf. [Tat67], Prop. 2.2.1) let $\mathbb{G}'$ be its Cartier dual and $T(\mathbb{G}')$ be the corresponding Tate module (cf. [loc.cit.], (1.2)/(2.4)). The latter is a free rank one $o$-module carrying an action of the absolute Galois group $G_L := G(\bar{L}/L)$ of $L$ which is given by a continuous character $\tau' : G_L \to o^\times$. Denote by $\mathbb{G}_m$ the formal multiplicative group over $o_{\mathbb{C}_p}$. There is a canonical Galois equivariant isomorphism of $o$-modules

$$T(\mathbb{G}') \cong \mathrm{Hom}_{o_{\mathbb{C}_p}}(\mathbb{G} \otimes_{o_L} o_{\mathbb{C}_p}, \mathbb{G}_m) \tag{14}$$

(cf. [loc.cit.], §4) where on the right-hand side, $G_L$ acts coefficientwise on formal power series over $o_{\mathbb{C}_p}$ and the $o$-module structure comes by functoriality from the formal $o$-module $\mathbb{G}$. Choose once and for all an $o$-module generator $t'$ for $T(\mathbb{G}')$ and denote by

$$F_{t'}(Z) = \omega Z + ... \in o_{\mathbb{C}_p}[[Z]]Z$$

the corresponding homomorphism of formal groups. By (14) the torsion points of $\mathbb{G}'$ are contained in the extension $L_\infty$ whence the "period" $\omega$ lies in the closure of $L_\infty$ (cf. [ST02], Appendix). Also, $\sigma.F_{t'} = \tau'(\sigma)F_{t'}$ and therefore $\omega^\sigma = \tau'(\sigma)\omega$ for all $\sigma \in G_L$. Choose $\omega_n \in L_\infty$ such that $|\omega_n/\omega - 1| < p^{-n}$ for all $n \in \mathbb{N}_0$ and put $L_n := L(\omega_n)$. We may assume that $L_n \subseteq L_{n+1}$ and that each $L_n$ is Galois. Define a formal power series as

$$h_n(Z) := \exp_{\mathbb{G}}((\omega_n/\omega)\log_{\mathbb{G}}(Z)) \in \mathbb{C}_p[[Z]]Z.$$

The generic fibre of the formal scheme Spf $o[[X]]$ underlying $\mathbb{G}$ equals the open unit disc around zero $\mathbf{B} \in Rig(L)$ (cf. [Ber], Remarque (0.2.6)) which, by functoriality, becomes in this way an $o$-module object in $\mathcal{G}_L$. We denote the action $o \times \mathbf{B} \to \mathbf{B}_0$ by $(g, z) \mapsto [g].z$. Let $\{\mathbf{B}(r_n)\}_{n\in\mathbb{N}}$ be the increasing admissible open affinoid covering of $\mathbf{B}$ by closed discs of radii $r_n := r^{1/q^{en}}$ where $r := p^{-q/e(q-1)}$. With the induced structure each disc $\mathbf{B}(r_n)$ becomes an $o$-module object in $\mathcal{G}_L$.

For each complete intermediate extension $L \subseteq F \subseteq \mathbb{C}_p$ there is an increasing admissible open affinoid covering $\{\hat{o}_{F,n}\}_{n\in\mathbb{N}}$ of the character variety $\hat{o}_F \in \mathcal{G}_F$ compatible with base extension (cf. [ST01], §3). By the uniformization theorem ( [ST01], Thm. 3.6) there is a compatible family of isomorphisms in $\mathcal{G}_{\mathbb{C}_p}$

$$\mathbf{B}(r_n)\,\hat{\otimes}_L\mathbb{C}_p \xrightarrow{\cong} \hat{o}_n\,\hat{\otimes}_L\mathbb{C}_p.$$

They glue to a group isomorphism $\kappa$ between $\mathbf{B}\,\hat{\otimes}_L\mathbb{C}_p$ and $\hat{o}\,\hat{\otimes}_L\mathbb{C}_p$ given on $\mathbb{C}_p$-valued points by

$$\kappa_z(g) = 1 + F_{t'}([g].z),\ z \in \mathbf{B}(\mathbb{C}_p),\ g \in o.$$

Remark: According to the remark following [ST01], Corollary 3.7 the isomorphism $\kappa$ descends from $\mathbb{C}_p$ to the closure of the field $L_\infty$ which clearly contains wild ramification. Furthermore, if $L \neq \mathbb{Q}_p$ then, according to [ST01], Lemma 3.9, the twisted form $\hat{o}$ of $\mathbf{B}$ cannot be trivialized over any discretely valued extension of $L$. In contrast, the main result of [Duc], Thm. (3.6) shows that any form of the space $\mathbf{B}$ with respect to a *tamely ramified* finite extension is already trivial.

**Lemma 5.5.** *For all $n \in \mathbb{N}$ the power series $h_n$ is a rigid analytic group automorphism of $\mathbf{B}(r_n)_{\mathbb{C}_p}$.*

*Proof:* We give the details of a proof sketched in [ST]. Let $\mathbb{G}(X,Y) \in o[[X,Y]]$ be the formal group law of $\mathbb{G}$. By basic properties of $\exp_{\mathbb{G}}$ and $\log_{\mathbb{G}}$ one computes that $h_n(Z)$ equals

$$\exp_{\mathbb{G}}(\mathbb{G}_a(\log_{\mathbb{G}}(Z), (\omega_n/\omega - 1)\log_{\mathbb{G}}(Z))) = \mathbb{G}(Z, \exp_{\mathbb{G}}((\omega_n/\omega - 1)\log_{\mathbb{G}}(Z))) \tag{15}$$

as formal power series over $\mathbb{C}_p$ where $\mathbb{G}_a$ denotes the formal additive group. By [ST01], Lemma 3.2 we have $[p^n].\mathbf{B}(r_n) = \mathbf{B}(r)$ whence

$$p^n \log_{\mathbb{G}}(\mathbf{B}(r_n)) = \log_{\mathbb{G}}([p^n].\mathbf{B}(r_n)) = \log_{\mathbb{G}}(\mathbf{B}(r)) = \mathbf{B}(r)$$

where the last identity follows from [Lan90], Lem. §8.6.4. Hence, on $\mathbf{B}(r_n)$ we have the composite of the rigid analytic functions

$$\mathbf{B}(r_n)_{\mathbb{C}_p} \xrightarrow{\log_{\mathbb{G}}} p^{-n}\mathbf{B}(r)_{\mathbb{C}_p} \xrightarrow{(\omega_n/\omega)-1} \mathbf{B}(r)_{\mathbb{C}_p} \xrightarrow{\exp_{\mathbb{G}}} \mathbf{B}(r)_{\mathbb{C}_p}. \tag{16}$$

Using that the group law $\mathbb{G}$ is defined over $o$ it follows that $h_n$ is a rigid analytic function on $\mathbf{B}(r_n)_{\mathbb{C}_p}$. Applying the same reasoning to the formal inverse $h_n^{-1}(Z) = \exp_{\mathbb{G}}((\omega/\omega_n)\log_{\mathbb{G}}(Z))$ shows that $h_n$ is a rigid automorphism of $\mathbf{B}(r_n)_{\mathbb{C}_p}$. It is clear from the definition of $h_n$ that it respects the Lubin-Tate group structure on $\mathbf{B}(r_n)_{\mathbb{C}_p}$. $\square$

**Lemma 5.6.** *The group isomorphism*

$$\kappa \circ h_n : \mathbf{B}(r_n) \hat{\otimes}_L \mathbb{C}_p \xrightarrow{\cong} \hat{o}_n \hat{\otimes}_L \mathbb{C}_p$$

*descends to the field $L_n$.*

*Proof:* Denote by $B_n := \mathcal{O}(\mathbf{B}(r_n))$ and $A_n := \mathcal{O}(\hat{o}_n)$ the $L$-affinoid algebras of $\mathbf{B}(r_n)$ and $\hat{o}_n$ respectively. We have obvious actions of the Galois group $G_n := Gal(\bar{L}/L_n)$ on $\mathbf{B}(r_n)_{\mathbb{C}_p}, B_n \hat{\otimes}_L \mathbb{C}_p$ and $\mathrm{Aut}_{\mathbb{C}_p-alg}(B_n \hat{\otimes}_L \mathbb{C}_p)$, the group of $\mathbb{C}_p$-algebra automorphisms of $B_n \hat{\otimes} \mathbb{C}_p$. Given $\sigma \in G_n$ and a $\mathbb{C}_p$-valued point $z$ of $\mathbf{B}(r_n)$ we find

$$h_n^{-1}(\sigma(h_n(z))) = \exp_{\mathbb{G}}((\omega/\omega^{\sigma})\log_{\mathbb{G}}(z)) = \exp_{\mathbb{G}}(\tau'(\sigma)^{-1}\log_{\mathbb{G}}(z)) = [\tau'(\sigma)^{-1}].z$$

where the last identity comes from [Lan90], Lemma §6.2. It follows for each $\sigma \in G_n$ that

$$\sigma.h_n^{\sharp} = h_n^{\sharp} \circ [\tau'(\sigma)^{-1}]$$

in $\mathrm{Aut}_{\mathbb{C}_p}(B_n \hat{\otimes}_L \mathbb{C}_p)$. By [ST01], Corollary 3.8 the Galois cocycle

$$G_L \to \mathrm{Aut}_{\mathbb{C}_p-alg}(B_n \hat{\otimes}_L \mathbb{C}_p), \ \sigma \mapsto [\tau'(\sigma)^{-1}]$$

gives the descent datum for the twisted form $\kappa^{\sharp} : A_n \hat{\otimes}_L \mathbb{C}_p \xrightarrow{\cong} B_n \hat{\otimes}_L \mathbb{C}_p$. According to the usual formalism of Galois descent (cf. [KO74], §9) we may therefore conclude that the $\mathbb{C}_p$-algebra automorphism

$$h_n^{\sharp} \circ \kappa^{\sharp} : A_n \hat{\otimes}_L \mathbb{C}_p \xrightarrow{\cong} B_n \hat{\otimes}_L \mathbb{C}_p$$

is $G_n$-equivariant. By the existence of topological $L$-bases (cf. [Sch02], Prop. 10.1) for $A_n$ and $B_n$ taking $G_n$-invariants and applying Tate's theorem $\mathbb{C}_p^{G_n} = L_n$ (cf. [Tat67], Prop. 3.1.8) yields the claim. $\square$

Remark: Using that $F_{t'}(Z) = \exp(\omega \log_{\mathbb{G}}(Z))$ as power series over $\mathbb{C}_p$ (cf. [ST01], Sect. 4) the isomorphism $\kappa \circ h_n$ of the preceding proposition is given on $\mathbb{C}_p$-points $z$ of $\mathbf{B}(r_n)$ via

$$(\kappa \circ h_n)_z(g) = \exp(g\omega_n \log_{\mathbb{G}}(z)), \ g \in o.$$

We now consider for each $n$ the base extension $\hat{o}_{K,n}$ of $\hat{o}$ to $K$. By the previous proposition it is a twisted form of the Lubin-Tate group on $\mathbf{B}(r_n)$ trivialized over $K_n := KL_n$ by $\kappa \circ h_n$. Let $m = m(n) \geq 0$ such that $p^m$ equals the $p$-part of $e(K_n/K)$. By Thm. 4.7 there is a higher derivation $\partial'_n$ on $C := (gr^\bullet K)^{p^{-m}}[z]$ such that $\Psi_C$ exhibits $Pic(A_n) = Pic(gr^\bullet A_n)$ to be finite cyclic of order dividing $p^m$ where $A_n := \mathcal{O}(\hat{o}_{K,n})$. We exhibit a generator. The ideal sheaf defining the zero section $Sp\ K \to \hat{o}_{K,n}$ is an invertible sheaf (Lem. 4.2) lying in $Pic(\hat{o}_{K,n})$.

**Proposition 5.7.** *The ideal sheaf defining the zero section generates $Pic(\hat{o}_{K,n})$.*

*Proof:* Let $A := \mathcal{O}(\hat{o}_{K,n})$ and $B := \mathcal{O}(\mathbf{B}(r_n)_{K_n})$ and let $I \subseteq A$ and $J \subseteq B$ the ideals of definition for the zero sections. The trivialization $\kappa \circ h_n$ is a group homomorphism and therefore preserves the origin. Applying the functor $gr^\bullet$ we therefore obtain a commutative diagram of $gr^\bullet K_n$-modules

$$\begin{tikzcd}
gr^\bullet I \otimes_{gr^\bullet K} gr^\bullet K_n = gr^\bullet (I \otimes_K K_n) \arrow[r, "\simeq"] \arrow[d] & gr^\bullet J \arrow[d] \\
gr^\bullet A \otimes_{gr^\bullet K} gr^\bullet K_n = gr^\bullet (A \otimes_K K_n) \arrow[r, "\simeq"] \arrow[d] & gr^\bullet B \arrow[d] \\
gr^\bullet K_n \arrow[r, "="] & gr^\bullet K_n
\end{tikzcd}$$

which, by a previous argument, descends to $(gr^\bullet K)^{p^{-m}}$. Hence,

$$I \otimes_{gr^\bullet K} (gr^\bullet K)^{p^{-m}} = (z)$$

where $(z)$ equals the principal ideal generated by the parameter $z \in (gr^\bullet K)^{p^{-m}}[z] = C$. Since $(z)$ is a line bundle so is $gr^\bullet I$ by faithfully flat descent (cf. [KO74], Lem. I.3.6) whence $gr^\bullet I \in Cart(gr^\bullet A)$. With $D := D(gr^\bullet I) \in D(gr^\bullet A)$ we thus have $j(D) = div_C(z)$ in $D(C)$. By definition $\Phi_C$ then maps the class of $D$ to $\partial'(z)/z$. According to Prop. 4.6 the line bundle $gr^\bullet I$ then generates $Pic(gr^\bullet A)$. It remains to see that $I \mapsto gr^\bullet I$ via $Pic(A) \xrightarrow{\cong} Pic(gr^\bullet A)$. Now $A$ is a Zariski ring. The induced filtration on the finitely generated $A$-submodule $I$ is therefore a good filtration (cf. [LvO96], Thm. II.2.1.2) and $gr^\bullet I$ being projective in $gr^\bullet A$-gr implies that $I$ is a projective object in $A$-filt (cf. [LvO96], Prop. I.7.2.10). Thus, Cor. 2.4 completes the proof. $\square$

It follows immediately from [ST01], Lem. 3.10 that the ideal sheaf corresponding to the zero section $SpK \to \hat{o}_K$ is an invertible sheaf in $Pic(\hat{o}_K)$ which has order 1 if $L = \mathbb{Q}_p$ and order $\neq 1$ if $L \neq \mathbb{Q}_p$.

**Proposition 5.8.** *Let $L \neq \mathbb{Q}_p$. The ideal sheaf defining the zero section $Sp\ K \to \hat{o}_K$ is not torsion in $Pic(\hat{o}_K)$.*

*Proof:* This is a generalization of the proof of [ST01], Lem. 3.10. Abbreviate $A := \mathcal{O}(\hat{o}_K)$, $B := \mathcal{O}(\mathbf{B}_K)$ and $B_{\mathbb{C}_p} := \mathcal{O}(\mathbf{B}_{\mathbb{C}_p})$ and let $z \in B$ be a parameter. Assume for a contradiction that the ideal sheaf is torsion. If $I \subseteq A$ denotes the corresponding ideal of global sections there is $1 < m < \infty$ such that $I^m = (f)$ with some $f \in A$. Since the trivialization $\kappa$ preserves the origin we have $I \mapsto (z)$ via $Pic(A) \to Pic(B_{\mathbb{C}_p})$ whence $(f) \mapsto (z^m)$. Now consider $f$ as a rigid $K$-analytic map $\hat{o}_K \to \mathbb{A}^1_K$ into the affine line over $K$. The composite $\mathbf{B}_{\mathbb{C}_p} \xrightarrow{\kappa} \hat{o}_{\mathbb{C}_p} \xrightarrow{f} \mathbb{A}^1_{\mathbb{C}_p}$ is then given by a power series $F$ generating the ideal $(z^m)$ of $B$ whence $F(z) = az^m(1 + b_1 z + b_2 z^2 + ...)$ with $a \in \mathbb{C}_p^\times$, $b_i \in o_{\mathbb{C}_p}$. Since our arguments will only depend on $K$ being discretely valued we may pass to a finite extension

of $K$ and therefore assume $x \in K$ with $|x| = |a|$. Passing to $x^{-1}f$ we see that $F$ induces a rigid $\mathbb{C}_p$-analytic map $\mathbf{B}_{\mathbb{C}_p} \to \mathbf{B}_{\mathbb{C}_p}$ being the union over $\mathbb{C}_p$-affinoid maps $\mathbf{B}(r_n)_{\mathbb{C}_p} \to \mathbf{B}(r_n)_{\mathbb{C}_p}$. Hence, $f$ is in fact a rigid $K$-analytic map

$$f : \hat{o}_K \to \mathbf{B}_K$$

equal to the union of $K$-affinoid maps $f_n : \hat{o}_{K,n} \to \mathbf{B}(r_n)_K$. The latter are induced by functions $f_n \in \mathcal{O}(\hat{o}_{K,n})$ generating $I^m\mathcal{O}(\hat{o}_{K,n})$, the $m$-th power of the ideal defining the zero section $Sp\ K \to \hat{o}_{K,n}$. Denote by $f^\sharp$ and $f_n^\sharp$ the corresponding ring homomorphisms on global sections. Now fix $n$ and let $A_n := \mathcal{O}(\hat{o}_{K,n}),\ B_n := \mathcal{O}(\mathbf{B}(r_n)_{K_n})$. Since our arguments will only depend on the fact that $K \subseteq K_n$ is finite we may assume that $a_n \in K_n$ such that $|a_n| = r_n$. Identifying $A \otimes_K K_n \simeq B_n$ via the group isomorphism $\kappa \circ h_n$ the map $f_n^\sharp \otimes_K K_n : B_n \to B_n$ is given by a power series in $B_n$ defining the $m$-th power of the zero section $Sp\ K \to \mathbf{B}(r_n)_{K_n}$. Hence $f_n^\sharp \otimes_K K_n$ equals the map $(a_n^{-1}z) \mapsto (a_n^{-1}z)^m\epsilon$ with suitable $\epsilon \in B_n^\times$. Since $|a_n^{-1}z| = 1$ this is an isometry with associated graded map

$$gr^\bullet\,(f_n^\sharp \otimes_K K_n) : \sigma(a_n^{-1}z) \mapsto \sigma(a_n^{-1}z)^m\sigma(\epsilon)$$

and $\sigma(\epsilon) \in (gr^\bullet K_n)^\times$. Clearly, the homomorphism $gr^\bullet\,(f_n^\sharp \otimes_K K_n)$ is finite free of rank $m$ on the homogeneous basis elements $\sigma(a_n^{-1}z)^0, ..., \sigma(a_n^{-1}z)^{m-1}$. Hence, $f_n^\sharp \otimes_K K_n$ is finite free of rank $m$ (cf. [LvO96], Lem. I.6.4). By faithfully flat descent $A_n$ is therefore a finitely generated projective $B_n$-module of rank $m$ via $f_n^\sharp$. This shows the $\mathcal{O}_{\mathbf{B}}$-module $f_*(\hat{o}_K)$ to be a vector bundle of rank $m$ which must be trivial (cf. [Gru68], Remarque V.2.3). Since $\mathbf{B}_K$ is quasi-Stein over the discretely valued field $K$ the $K$-algebra $B$ is Fréchet-Stein in the sense of [**?**], Sect. 3. We therefore have the full subcategory of coadmissible $B$-modules at our disposal in which every object is endowed with a canonical topology and in which every morphism is strict. The $B$-module isomorphism $B^m \xrightarrow{\cong} A$ coming from taking global sections in the $\mathcal{O}_{\mathbf{B}}$-module $f_*(\hat{o}_K)$ and choosing a basis lies in this category and is therefore topological with respect to the canonical topology on both sides. By definition of the latter as a certain Fréchet topology this isomorphism must respect the subrings of bounded elements. All in all, we obtain that the restriction $f^\sharp : B^b \to A^b$ is finite free of rank $m$. Let $g \in A^b$ be nonzero and consider an integral equation of minimal degree $g^d b_d + g^{d-1}b_{d-1} + ... + b_0 = 0$ over $B^b$. Then $b_0 \neq 0$ and thus, by the Weierstrass preparation theorem (cf. [Bou98], VII§3.8 Prop. 6), $b_0$ has only finitely many zeroes on $\mathbf{B}_K$. Since $f$ has finite fibres (cf. [BGR84], Cor. 9.6.3/6) it follows that $g$ has only finitely many zeroes on $\hat{o}_K$. But according to (the proof of) [ST01], Lem. 3.9 the nonzero function given on $\mathbb{C}_p$-valued points via $\kappa_z \mapsto \kappa_z(1) - \kappa_z(0)$ lies in $A^b$ and has infinitely many zeroes, a contradiction. $\square$

We prove the main result of this section.

**Theorem 5.9.** *The group $Pic(\hat{o}_K)$ is pro-$p$ and topologically generated by the ideal sheaf defining the zero section $Sp\ K \to \hat{o}_K$. In particular, $Pic(\hat{o}_K) = 1$ if $L = \mathbb{Q}_p$ and $Pic(\hat{o}_K) = \mathbb{Z}_p$ otherwise.*

*Proof:* Let $A := \mathcal{O}(\hat{o}_K), A_n := \mathcal{O}(\hat{o}_{K,n})$ and $B_n = \mathcal{O}(\mathbf{B}(r_n)_{K_n})$. Let $I \subseteq A$ be the ideal of definition for the zero section and let $I_n := IA_n$. Consider the natural homomorphism

$$\pi : Pic(A) \longrightarrow \varprojlim_n Pic(A_n). \tag{17}$$

By our results so far the right-hand side is a projective limit of cyclic $p$-groups each having $I_n$ as a generator. It is therefore a pro-cyclic pro-$p$ group on the topological generator $\pi(I)$. Since $I \in Pic(A)$ has order 1 if $L = \mathbb{Q}_p$ and infinite

order otherwise it suffices to prove bijectivity of $\pi$. Surjectivity follows from Cor. 5.3. For injectivity we use an adaption of the argument given in [Gru68], Prop. V.3.2. Let $[P] \in \mathbf{P}(A)$ be an element in the kernel of (17). For each $n \in \mathbb{N}$ let $P_n := P \otimes_A A_n$ and let $x_n \in P_n$ be an $A_n$-module generator. If $n \geq m$ then $A_n \rightarrow A_m$ is injective and hence, so is $P_n \rightarrow P_m$. Since the image of $x_n$ under the map $P_n \rightarrow P_m$ generates $P_m$ we may write $x_n = a_m x_m$ with $a_m \in A_m^\times$. Let $U_n := 1 + F^{0+}A_n \subseteq A_n^\times$ where $F^{0+}A_n = \{a \in A_n : |a| < 1\}$. Taking Galois invariants in $B_n^\times = K_n^\times \times (1 + F^{0+}B_n)$ yields $A_n^\times = K^\times U_n$. Thus, we may assume that $a_n \in U_n$. The convex sets $V_n := U_n x_n \subseteq P_n$ satisfy $V_{n+1} \subseteq V_n$. Now for each $n \in \mathbb{N}$ the map $P_n \rightarrow P_{n-1}$ is a compact linear map between $p$-adic Banach spaces over the locally compact field $K_n$ (cf. [ST03], Lem. 6.1). We may therefore argue as in [Gru68], Prop. V.3.2 to obtain that $\cap_n V_n \neq \emptyset$. Any nonzero element in this intersection then generates $P$ whence $[P] = 0$. $\square$

Remark: The theorem answers a question raised by J. Teitelbaum (cf. [Tei], Question 2). As a special case it includes $Pic(\mathbf{B}_K) = 0$, a well-known theorem of M. Lazard (cf. [Laz62], Thm. 7.2) on the geometry of the one dimensional open unit disc over a discretely valued field.

By [ST01], §3 the algebra of global sections $A := \mathcal{O}(\hat{o}_K)$ is a Prüferian domain whence $SK_0(A) = 0$ by Lemma 2.2.

**Corollary 5.10.** *Rank and determinant gives canonical isomorphisms $K_0(\hat{o}_K) \simeq \mathbb{Z}$ if $L = \mathbb{Q}_p$ and $K_0(\hat{o}_K) \simeq \mathbb{Z} \oplus \mathbb{Z}_p$ otherwise.*

5.3. **General character varieties.** We wish to extend our results on the additive group to more general abelian locally $L$-analytic groups.

Let $Z$ be an abelian locally $L$-analytic group which is (topologically) finitely generated. Denote by $\mu \subseteq Z$ its torsion subgroup.

**Lemma 5.11.** *The group $\mu$ is finite. There exists $r \in \mathbb{N}_0$ and an isomorphism of locally $L$-analytic groups where $d := \dim_L Z$*

$$\mu \times o^d \times \mathbb{Z}^r \xrightarrow{\cong} Z.$$

*Proof:* By [Eme], Prop. 6.4.1 the inclusion of the unique maximal compact open subgroup $Z_0$ into $Z$ induces an isomorphism $Z_0 \times \mathbb{Z}^r \cong Z$ for some unique $r \in \mathbb{N}_0$. By the structure theorem on abelian profinite groups (e.g. [RV99], Cor. 1.24) $Z_0$ equals (as a topological group) the product over its Sylow subgroups. The unique $p$-Sylow subgroup $Z_0(p)$ contains any open standard subgroup of $Z_0$ (in the sense of [Bou72], Thm. III.7.3.4) whence is open. Hence, $\mu$ is finite. Moreover, $Z_0(p)$ is a topologically finitely generated abelian pro-$p$ group whence a finitely generated $\mathbb{Z}_p$-module. It therefore decomposes as a locally $\mathbb{Q}_p$-analytic group into a finite and an open subgroup isomorphic to $\mathbb{Z}_p^{d[L:\mathbb{Q}_p]}$. Given the induced locally $L$-analytic structure on the latter it identifies with the locally $L$-analytic group $o^d$. $\square$

For all upcoming basic notions from nonarchimedean functional analysis we refer to [Sch02]. Let $V$ be a Hausdorff locally convex $K$-vector space. Denote by $C^{an}(Z, V)$ the $K$-vector space of $V$-valued locally $L$-analytic functions on $Z$ and denote by $D(Z, K) := C^{an}(Z, K)'_b$ the algebra of $K$-valued locally $L$-analytic distributions on $Z$ ( [ST02], §2). Both spaces carry natural Hausdorff locally convex topologies making the convolution product in $D(Z, K)$ separately continuous. If $Z$ is compact $D(Z, K)$ is a nuclear Fréchet-Stein algebra (in the sense of [ST03], §3).

We introduce the following generalization (due to M. Emerton, cf. [Eme], (6.4)) of the rigid analytic character variety $\hat{o}$ studied in the preceding section. For each $X \in Rig(K)$ we denote by $\hat{Z}(X)$ the group of abstract group homomorphisms $Z \to \mathcal{O}(X)^\times$ with the property that for each admissible open affinoid subspace $U \subseteq X$ the map $Z \to \mathcal{O}(X)^\times \xrightarrow{res} \mathcal{O}(U)$ lies in $C^{an}(Z, \mathcal{O}(U))$. Here, $res$ is the natural restriction map $\mathcal{O}(X) \to \mathcal{O}(U)$ restricted to $\mathcal{O}(X)^\times$. This defines a contravariant functor $\hat{Z}_K : Rig(K) \to Ab$. It is representable by a smooth rigid $K$-analytic group $\hat{Z}_K$ on a quasi-Stein space. If $Z = (o, +)$ then $\hat{Z}_K$ coincides with the character group $\hat{o}\hat{\otimes}_L K$ introduced before.

**Proposition 5.12.** *Keep the notation of the lemma. One has* $\pi_0(\hat{Z}_K) \leq \#\mu$ *and* $\dim \hat{Z}_K = d + r$. *There is an isomorphism of rigid $K$-analytic groups*

$$\hat{\mu}_K \times_K \hat{o}_K^d \times_K \mathbb{G}_m^r \xrightarrow{\cong} \hat{Z}_K.$$

*Proof:* This is a modest refinement of the argument given in [Eme], Prop. 6.4.5. and follows from the above lemma like this. Denote for $m \geq 1$ by $\mu_m$ the rigid $K$-analytic group equal to the $m$-th roots of unity. According to [loc.cit.] and [Buz04], Lemma 2 we have $\hat{\mathbb{Z}}_K = \mathbb{G}_m$ and and $\widehat{\mathbb{Z}/m\mathbb{Z}} = \mu_m$. Associating $Z \mapsto \hat{Z}_K$ converts direct products into fibre products over $K$. By the preceding lemma we therefore have a group isomorphism in $Rig(K)$

$$\hat{\mu} \times_K \hat{o}_K^d \times_K \mathbb{G}_m^r \xrightarrow{\cong} \hat{Z}_K.$$

It remains to see the formulas on connected components and dimension. Being finite étale the space $\hat{\mu}_K$ is a finite disjoint union over points $Sp\ K_i,\ i = 1, ..., s$ where each $K \subseteq K_i$ is a finite field extension. Since each $K_i \times_K \hat{o}_K^d$ is connected and $\mathbb{G}_m^r$ is geometrically connected it follows from [Koh08], Prop. A.7 that their fibre product is connected. Hence, $\pi_0(\hat{Z}_K) = s$. Since $\pi_0(\widehat{\mathbb{Z}/m\mathbb{Z}}_K) \leq m$ for $m \geq 1$ we obtain $s \leq \#\mu$ by taking fibre products. Finally, $\dim \hat{o}_K = 1$ as remarked earlier from which the dimension formula follows. $\square$

Remark: As mentioned in the introduction the main example of a group like $Z$ is the group $\mathbb{T}(L)$ of $L$-rational points of a linear algebraic torus $\mathbb{T}$ defined over $L$ ( note that topologically finite generation for the split part is immediate and follows for the anisotropic part by compactness, cf. [BT65], Cor. §9.4). Denote by $\mathrm{Rep}_K^{esa}(T)$ and $Coh(\hat{T}_K)$ the abelian categories of essentially admissible locally $L$-analytic $T$-representations over $K$ (cf. [Eme], (6.4)) and coherent modules sheaves on $\hat{T}_K$ respectively. The global section functor followed by "passage to the strong continuous dual" induces an anti-equivalence

$$Coh(\hat{T}_K) \xrightarrow{\cong} \mathrm{Rep}_K^{esa}(T)$$

under which the structure sheaf corresponds (at least if $r = 0$) to the regular representation. As a next step it is therefore desirable to have a description of the locally free sheaves in $Coh(\hat{T}_K)$.

5.4. **Vector bundles.** Assuming $\dim_L Z \leq 1$ we compute the Picard group and the Grothendieck group of $\hat{Z}_K$.

By Prop. 5.12 we have a morphism of locally $L$-analytic groups

$$\iota : \mu \times o^d \to \mu \times o^d \times \mathbb{Z}^r \longrightarrow Z,\ (g, h) \mapsto (g, h, 0)$$

inducing the natural projection map

$$\hat{\iota} : \hat{Z}_K = \hat{\mu}_K \times_K \hat{o}_K^d \times_K \mathbb{G}_m^r \to \hat{\mu}_K \times_K \hat{o}_K^d$$

**Proposition 5.13.** *Let $d \leq 1$. The induced maps on Grothendieck and Picard groups are bijective.*

*Proof:* The space $\hat{\mu}_K$ is finite étale. Since it will become clear that our arguments do not depend on finite field extensions of $K$ we may in the following assume, by additivity of $K_0$ on connected components, that $\mu = 1$. In case $d = 0$ we have the $Pic(K) = Pic(\mathbb{G}_m^r) = 0$ and $K_0(K) = K_0(\mathbb{G}_m^r) = \mathbb{Z}$ (cf. [FvdP04], 6.3) whence the result follows. Assume $d = 1$ in the following and consider $K_0$. Recall our specified covering $\{\hat{o}_{n,K}\}_{n\in\mathbb{N}}$ of $\hat{o}_K$ from section 5.2 where we put $\hat{o}_{n,K} := \hat{o}_n \otimes_L K$. Let $z$ be the parameter on the analytic torus $\mathbb{G}_m$ defining the unit section and consider the admissible open covering of $\mathbb{G}_m$ given by the affinoid subdomains $|p|^n \leq z \leq |p|^{-n}$ as $n$ tends to infinity. Taking fibre products produces an increasing admissible open affinoid covering $\{\hat{Z}_{n,K}\}_{n\in\mathbb{N}}$ of $\hat{Z}_K = \hat{o}_K \times_K \mathbb{G}_m^r$ realising its quasi-Stein property. By construction, $\hat{\iota}$ gives rise, for each $n \in \mathbb{N}$, to the natural projection between $K$-affinoids

$$\hat{\iota}_n : \hat{Z}_{n,K} \longrightarrow \hat{o}_{n,K}.$$

By Lemma 4.2 we may apply Lemma 5.16 and obtain that the associated map on $K_0$-groups

$$\hat{\iota}_n^* : K_0(\hat{o}_{n,K}) \xrightarrow{\cong} K_0(\hat{Z}_{n,K}) \tag{18}$$

is bijective. We now consider the diagram

$$\begin{array}{ccc} K_0(\hat{o}_K) & \xrightarrow{=} & K_0(\hat{o}_K) \\ {\scriptstyle \hat{\iota}^*}\big\downarrow & & \big\downarrow {\scriptstyle \varprojlim_n res_n^*} \\ K_0(\hat{Z}_K) & \xrightarrow{\psi} & \varprojlim_n K_0(\hat{o}_{n,K}) \end{array} \tag{19}$$

where $res_n : \mathcal{O}(\hat{o}_K) \to \mathcal{O}(\hat{o}_{n,K})$ equals the restriction map and $\psi$ equals the composite of the maps

$$\varprojlim_n res_n^* : K_0(\hat{Z}_K) \to \varprojlim_n K_0(\hat{Z}_{n,K}) \; , \; \varprojlim_n (\hat{\iota}_n^*)^{-1} : \varprojlim_n K_0(\hat{Z}_{n,K}) \to \varprojlim_n K_0(\hat{o}_{n,K}).$$

To start with, the right vertical arrow in (19) is bijective. Indeed, by the last remark of section 5.2 it suffices to check this on Picard groups which was done in the proof of theorem 5.9.

We claim that $\psi$ is also bijective. The commutativity of the last diagram is readily checked and implies

$$\psi \circ \hat{\iota}^* \circ (\varprojlim_n res_n^*)^{-1} = id.$$

To prove that $\hat{\iota}^* \circ (\varprojlim_n res_n^*)^{-1} \circ \psi = id$ pick $[P] \in K_0(\mathcal{O}(\hat{Z}_K)) = K_0(\hat{Z}_K)$ and take $[M] \in K_0(\mathcal{O}(\hat{o}_K)) = K_0(\hat{o}_K)$ such that

$$(\varprojlim_n res_n^*)([M]) = \psi([P]). \tag{20}$$

We show that $\hat{\iota}^*([M]) = [P]$. First, the maps (1) $P \to \varprojlim_n P \otimes_{\mathcal{O}(\hat{Z}_K)} \mathcal{O}(\hat{Z}_{n,K})$ and (2) $M \otimes_{\mathcal{O}(\hat{o}_K)} \mathcal{O}(\hat{Z}_K) \to \varprojlim_n M_n \otimes_{\mathcal{O}(\hat{o}_{n,K})} \mathcal{O}(\hat{Z}_{n,K})$ are bijections. Indeed, since $\hat{Z}$ is quasi-Stein the algebra $\mathcal{O}(\hat{Z}_K)$ is a Fréchet-Stein algebra in the sense of [ST03], sect. 3. Now $P$ is a finitely generated projective $\mathcal{O}(\hat{Z}_K)$-module whence finitely presented. By [ST03], Cor.3.4 it is therefore a coadmissible module and [loc.cit.], Cor. 3.1 gives (1). Secondly

$$\mathcal{O}(\hat{Z}_K) = \mathcal{O}(\hat{o}_K)\hat{\otimes}_K \mathcal{O}(\mathbb{G}_m^r) \; , \; \mathcal{O}(\hat{Z}_{n,K}) = \mathcal{O}(\hat{o}_{n,K})\hat{\otimes}_K \mathcal{O}(\mathbb{G}_{m,n}^r)$$

as algebras and for each $n \in \mathbb{N}$. Hence (2) follows from associativity of the (completed) tensor product and the Fréchet-Stein property of $\mathcal{O}(\mathbb{G}_m^r)$. Next

$$[M_n \otimes_{\mathcal{O}(\hat{o}_{n,K})} \mathcal{O}(\hat{Z}_n)] = \hat{\iota}_n^*[M \otimes_{\mathcal{O}(\hat{o}_K)} \mathcal{O}(\hat{o}_{n,K})] = [P \otimes_{\mathcal{O}(\hat{Z}_K)} \mathcal{O}(\hat{Z}_{n,K})],$$

the first equality by definition of $\hat{\iota}_n$ and the second because of (20). Applying $\varprojlim_n$ on both sides and using (1) and (2) yields $\hat{\iota}^*([M]) = [P]$ as desired. Since $K_0(\mathcal{O}(\hat{Z}_K))$ is generated by such $[P]$ the injectivity of $\psi$ follows. Now the bijection $K_0(\hat{\mu}_K \times_K \hat{o}_K^d) \xrightarrow{\cong} K_0(\hat{Z}_K)$ is clearly compatible with rank and determinant mappings. Hence, an argument analogous to the one in the proof of Lem. 2.3 implies the bijectivity on Picard groups. □

Summarizing our results so far yields the

**Theorem 5.14.** *Let $d \leq 1$. Then rank and determinant induces an isomorphism*

$$\mathrm{rk} \oplus \det : K_0(\hat{Z}_K) \xrightarrow{\cong} \mathbb{Z}^{\#\pi_0(\hat{Z}_K)} \oplus Pic(\hat{Z}_K)$$

*where $\#\pi_0(\hat{Z}_K) \leq \#\mu$ and $Pic(\hat{Z}_K) = Pic(\hat{\mu}_K \times_K \hat{o}_K^d)$ canonically. In particular, $Pic(\hat{Z}_K) = 0$ unless $L \neq \mathbb{Q}_p$ and $d = 1$ where $Pic(\hat{Z}_K) = \mathbb{Z}_p^{\#\pi_0(\hat{Z}_K)}$ canonically.*

As a last result we clarify the relation between $K_0(\hat{Z}_K) = K_0(\mathcal{O}(\hat{Z}_K))$ and the group $K_0(D(Z,K))$ associated to the abelian distribution algebra $D(Z,K)$ of $Z$. It is immediate from Proposition 5.12 that $\hat{Z}$ is *strictly $\sigma$-affinoid* (in the sense of [Eme], Def. 2.1.17). By the remarks following [loc.cit.], Def. 2.1.18 the $K$-algebra of global sections $\mathcal{O}(\hat{Z}_K)$ has a natural Hausdorff locally convex topology which makes it a nuclear Fréchet algebra. In this situation the Fourier homomorphism

$$F : D(Z,K) \longrightarrow \mathcal{O}(\hat{Z}),\ \lambda \mapsto F_\lambda \tag{21}$$

is a continuous injective algebra homomorphism with dense image and bijective if $Z$ is compact ( [Eme], Prop. 6.4.6/ [ST01], Thm. 2.2). It is given by $F_\lambda(z) = \lambda(\kappa_z)$ where $\kappa_z \in C^{an}(Z, \mathbb{C}_p)$ denotes the locally $L$-analytic character $Z \to \mathbb{C}_p^\times$ associated to $z \in \hat{Z}_K(\mathbb{C}_p)$.

**Corollary 5.15.** *Let $d \leq 1$. The map $F$ induces bijections on Picard and Grothendieck groups.*

*Proof:* The closed embedding of locally $L$-analytic groups $\iota$ induces a topological embeddding

$$\iota_* : D(\mu \times o^d, K) \to D(Z,K)$$

(cf. [Koh07], Prop.1.1.2). We obtain a commutative diagram of abelian groups

$$\begin{array}{ccc} K_0(D(\mu \times o^d, K)) & \xrightarrow{F^*} & K_0(\hat{\mu}_K \times_K \hat{o}_K^d) \\ {\scriptstyle (\iota_*)^*}\downarrow & & \downarrow{\scriptstyle \hat{\iota}^*} \\ K_0(D(Z,K)) & \xrightarrow{F^*} & K_0(\hat{Z}) \end{array}$$

where the upper horizontal and the right vertical arrow are bijective according to compactness of $\mu \times o^d$ and Prop. 5.13 respectively. By a previous argument we may assume that $\mu = 1$. Choosing indeterminates $z_1, ..., z_r$ the map $\iota_*$ becomes the natural inclusion

$$\iota_* : D(o^d, K) \to D(o^d, K)[z_1^{\pm}, ..., z_r^{\pm}] = K[\mathbb{Z}^r]\hat{\otimes}_K D(o^d, K) = D(Z,K).$$

Here, $K[.]$ refers to the ordinary group algebra over $K$ and the last equality comes from (cf. [ST05], Prop. A.3). If $d = 0$ the result follows. Furthermore, the ring $D(o,K)$ is a Prüferian domain (cf. [ST01], §3) whence Lem. 2.1 yields the bijectivity of $(\iota_*)^*$ in case $d = 1$. The case for Picard groups is similar. □

We finish this work by justifying the bijection (18) used in the proof of Prop. 5.15.

**Lemma 5.16.** *Let $A$ be a reduced affinoid $K$-algebra such that $gr^\bullet A$ is noetherian regular. Let $B_n$ denote the $K$-affinoid algebra corresponding to the affinoid subdomain of $\mathbb{G}_m^r$ given by*

$$|p|^n \leq z_i \leq |p|^{-n}$$

*for $i = 1, ..., r$ and $n \in \mathbb{N}_0$. The natural ring homomorphism $A \stackrel{x \mapsto x \otimes 1}{\longrightarrow} A \hat{\otimes}_K B_n$ induces an isomorphism $K_0(A) \xrightarrow{\cong} K_0(A \hat{\otimes}_K B_n)$.*

*Proof:* Since $K$ is discretely valued the spectral filtration on $A$ is quasi-integral (in the sense of [ST03], Sect. 1). We therefore have the obvious analogue of a strong $\mathbb{Z}$-gradation on $gr^\bullet A$ (cf. [LvO96], I.4.1) whence $gr^0 A$ and $F^0 A$ are noetherian regular (cf. [LvO96], Prop. II.5.1.12). After these remarks let $\varphi$ denote the homomorphism

$$A \to A \hat{\otimes}_K B_n = A\langle p^n z_1^{\pm 1}, ..., p^n z_r^{\pm 1}\rangle.$$

We first treat the case $n = 0$. The spectral norm filtration on the $K$-affinoid algebra $A \hat{\otimes}_k B_0$ gives (cf. [Gru68], Theorem IV.5.1)

$$gr^\bullet (A \hat{\otimes}_K B_0) = (gr^\bullet A)[z_1^{\pm 1}, ..., z_r^{\pm 1}].$$

Hence, according to [LvO96], Prop. II.2.2.1, the filtration on the ring $A \hat{\otimes}_K B_0$ is Zariskian. The graded ring $gr^\bullet (A \hat{\otimes}_K B_0)$ is noetherian regular (cf. [MR87], Thm. 7.5.3). According to Lem. 2.3 it now suffices to see that

$$gr^\bullet \varphi : gr^\bullet A \longrightarrow (gr^\bullet A)[z_1^{\pm 1}, ..., z_r^{\pm 1}]$$

induces an isomorphism on $K_0$-groups. Since $gr^\bullet A$ is noetherian regular this is included in Lemma 2.1. Now let $n > 0$ be arbitrary. Composing $\varphi$ with the restriction map to the annulus $z_i = 1, i = 1, ..., r$ we obtain $A \to A\langle z_1^{\pm 1}, ..., z_r^{\pm 1}\rangle$. By the case $n = 0$ this induces an isomorphism on $K_0$-groups whence $K_0(\varphi)$ is injective. Again consider the spectral norm filtration on $A \hat{\otimes}_K B_n$. It follows from [BGR84], 6.2.3 that $F^0(A \hat{\otimes}_K B_n) = (F^0 A) \hat{\otimes}_{o_K} (F^0 B_n)$ where $o_K \subseteq K$ denotes the valuation ring. By the analogue of [Con06], Thm. 4.2.7 for formal schemes the ring $F^0(A \hat{\otimes}_K B_n)$ is therefore noetherian regular. Furthermore, $gr^0(A \hat{\otimes}_K B_n)$ is seen to equal as $gr^0 A$-algebra the tensor product over $gr^0 A$ of $r$ copies of the algebra $(gr^0 A)[x, y]/(xy)$ ($x, y$ two variables). We therefore obtain a surjection

$$K_0(gr^0 A) \to K_0(gr^0(A \hat{\otimes}_K B_n) \simeq K_0(F^0(A \hat{\otimes}_K B_n)) \to K_0(A \hat{\otimes}_K B_n) \tag{22}$$

where the first surjection comes from the lemma below, the second isomorphism from [Bas68], Prop. 4.IX.1.3 and the final surjection from Quillen's localization sequence (cf. [Qui73], Thm. 12.4.9). Composing this map with

$$K_0(A) \xrightarrow{\cong} K_0(gr^\bullet A) \xrightarrow{\cong} K_0(gr^0 A)$$

(Lemma 2.3 and [MR87], Cor. 12.3.5) and checking that the composite equals $K_0(\varphi)$ we are done. □

The following lemma was used in the preceding proof.

**Lemma 5.17.** *Let $R$ be a commutative noetherian regular ring and let $A^i := R[t_{i1}, t_{i2}]/t_{i1}t_{i2}$ with variables $t_{i1}, t_{i2}$ for $i = 1, ..., r$. Let $A = A^1 \otimes_R \cdots \otimes_R A^r$. Then the natural map $K_0(R) \to K_0(A)$ is surjective.*

*Proof:* This is an adaption of the argument given in [HL00], Prop. 2.2. Let $M \in \mathbf{P}(A)$. Let $\sigma : A \to R$ be the zero section with kernel $I$ and put $N := \sigma^*(M)$. We show that $[N] \in K_0(R)$ maps to $[M] \in K_0(A)$ thus proving the lemma. We will assume $r = 2$ since the general case follows along the same lines but with more

notation. We put $A^0 := R[x_1, x_2]/x_1x_2, A^1 := R[y_1, y_2]/y_1y_2$ with indeterminates $x_i, y_i$ and identify $A = A^0 \otimes_R A^1$. Consider the exact sequence of free $R$-modules

$$0 \to A^0 \to A^0/x_1 \times A^0/x_2 \to A^0/x_1x_2 \to 0$$

where the maps are induced by the diagonal mapping and the map $(u, v) \mapsto u - v$. Tensoring this sequence over $R$ with the corresponding one for $A^1$ yields a short exact sequence of $A$-modules

$$0 \to A \to A_{11} \times A_{21} \times A_{12} \times A_{22} \to A/I \to 0$$

where we have abbreviated $A_{ij} := A/(x_i, y_j)$ and the maps are induced by the diagonal mapping and the map $(u, v, w, z) \mapsto u - v - w + z$. Since $M$ and $N$ are $A$-flat and $R$-flat respectively we obtain two short exact sequences of $A$-modules

$$\begin{array}{ccccccccc} 0 & \to & M & \to & M_{11} \times M_{21} \times M_{12} \times M_{22} & \to & M/I & \to 0, \\ 0 & \to & N \otimes_R A & \to & N_{11} \times N_{21} \times N_{12} \times N_{22} & \to & N \otimes_R A/I & \to 0 \end{array}$$

abbreviating $M_{ij} := M \otimes_A A_{ij}$ and $N_{ij} := N \otimes_R A_{ij}$. Now $M_{ij} \in \mathbf{P}(A_{ij})$ where $A_{ij}$ is a polynomial ring over $R$ in two variables. According to [MR87], Cor. 12.3.5 we have an equality in $K_0(A_{ij})$

$$[\sigma_{ij}^*(M_{ij}) \otimes_R A_{ij}] = [M_{ij}]$$

where $\sigma_{ij} : A_{ij} \to R$ denotes the zero section. Hence, there exists $n \in \mathbb{N}_0$ (independent of $ij$) and an isomorphism of $A_{ij}$-modules $\varphi_{ij}$ such that

$$\varphi_{ij} : (N \otimes_R A_{ij}) \oplus A_{ij}^n \xrightarrow{\cong} M_{ij} \oplus A_{ij}^n. \tag{23}$$

Put $\tilde{M} := M \oplus A^n$ such that $\tilde{N} := \sigma^*(\tilde{M}) = N \oplus R^n$. In obvious notation (23) gives an isomorphism $\varphi_{ij} : \tilde{N} \otimes_R A_{ij} \xrightarrow{\cong} \tilde{M}_{ij}$ for all $ij$. Furthermore, one checks that the induced isomorphism of $R$-modules

$$\sigma^*(\tilde{M}) = \tilde{N} = \sigma_{ij}^*(\tilde{N} \otimes_R A_{ij}) \xrightarrow{\sigma_{ij}^*(\varphi_{ij})} \sigma_{ij}^*(\tilde{M}_{ij}) = \tilde{M}/I\tilde{M}$$

coincides for each $ij$ with the natural map. We therefore have compatible isomorphisms between the middle and the right-hand side in the tilde versions of the above two exact sequences of $A$-modules. Hence, we obtain an isomorphism between the kernels which proves the lemma. $\square$

## References


[Bab81] K. Baba, *On p-radical descent of higher exponent*, Osaka J. Math. **18** (1981), no. 3, 725–748.

[Bas68] H. Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[Ber] P. Berthelot, *Cohomologie rigide et cohomologie rigide à supports propres (première partie)*, preprint. Available at: http://perso.univ-rennes1.fr/pierre.berthelot/.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis*, Springer-Verlag, Berlin, 1984.

[Bou72] N. Bourbaki, *Éléments de mathématique. Fasc. XXXVII. Groupes et algèbres de Lie. Chapitre II/III*, Hermann, Paris, 1972, Actualités Scientifiques et Industrielles, No. 1349.

[Bou98] ———, *Commutative algebra. Chapters 1–7*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998.

[BT65] A. Borel and J. Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 27, 55–150.

[Buz04] K. Buzzard, *On p-adic families of automorphic forms*, Modular curves and abelian varieties, Progr. Math., vol. 224, Birkhäuser, Basel, 2004, pp. 23–44.

[Con06] B. Conrad, *Relative ampleness in rigid geometry*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 4, 1049–1126.

[Duc] A. Ducros, *Toute forme modérément ramifiée d'un polydisque ouvert est triviale*, Preprint. Available at: http://math1.unice.fr/ ducros/pseudodisques.pdf.

[Eme] M. Emerton, *Locally analytic vectors in representations of locally $p$-adic analytic groups*, Preprint. To appear in: Memoirs of the AMS.

[Eme06] ———, *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms*, Invent. Math. **164** (2006), no. 1, 1–84.

[FvdP04] J. Fresnel and M. van der Put, *Rigid analytic geometry and its applications*, Progress in Mathematics, vol. 218, Birkhäuser Boston Inc., Boston, MA, 2004.

[Gla89] S. Glaz, *Commutative coherent rings*, Lecture Notes in Math., vol. 1371, Springer-Verlag, Berlin, 1989.

[Gro57] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221.

[Gru68] L. Gruson, *Fibrés vectoriels sur un polydisque ultramétrique*, Ann. Sci. École Norm. Sup. (4) **1** (1968), 45–89.

[Har75] R. Hartshorne, *On the De Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. (1975), no. 45, 5–99.

[Har77] ———, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Math., No. 52.

[HL00] U. Hartl and W. Lütkebohmert, *On rigid-analytic Picard varieties*, J. Reine Angew. Math. **528** (2000), 101–148.

[Jac75] N. Jacobson, *Lectures in abstract algebra. III*, Springer-Verlag, New York, 1975, Theory of fields and Galois theory, Graduate Texts in Math., No. 32.

[Kie67] R. Kiehl, *Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie*, Invent. Math. **2** (1967), 256–273.

[KO74] M.-A. Knus and M. Ojanguren, *Théorie de la descente et algèbres d'Azumaya*, Lecture Notes in Math., Vol. 389, Springer-Verlag, Berlin, 1974.

[Koh07] J. Kohlhaase, *Invariant distributions on $p$-adic analytic groups*, Duke Math. J. **137** (2007), no. 1, 19–62.

[Koh08] ———, *Lubin-Tate and Drinfel'd bundles*, Münster SFB-Preprint 505 (2008).

[Lan90] S. Lang, *Cyclotomic fields I and II*, second ed., Graduate Texts in Mathematics, vol. 121, Springer-Verlag, New York, 1990, With an appendix by Karl Rubin.

[Laz62] M. Lazard, *Les zéros des fonctions analytiques d'une variable sur un corps valué complet*, Inst. Hautes Études Sci. Publ. Math. (1962), no. 14, 47–75.

[LvO96] H. Li and F. van Oystaeyen, *Zariskian filtrations*, $K$-Monographs in Mathematics, vol. 2, Kluwer Academic Publishers, Dordrecht, 1996.

[Mat86] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986.

[MR87] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Pure and Applied Mathematics (New York), John Wiley & Sons Ltd., Chichester, 1987.

[Qui73] D. Quillen, *Higher algebraic $K$-theory. I*, Algebraic $K$-theory, I: Higher $K$-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.

[Rus70] P. Russell, *Forms of the affine line and its additive group*, Pacific J. Math. **32** (1970), 527–539.

[RV99] D. Ramakrishnan and R. J. Valenza, *Fourier analysis on number fields*, Graduate Texts in Mathematics, vol. 186, Springer-Verlag, New York, 1999.

[Sch02] P. Schneider, *Nonarchimedean functional analysis*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002.

[Ser02] J.-P. Serre, *Galois cohomology*, english ed., Springer Monographs in Math., Springer-Verlag, Berlin, 2002.

[ST] P. Schneider and J. Teitelbaum, *$p$-adic Fourier theory: Additional notes*, unpublished.

[ST01] ———, *$p$-adic Fourier theory*, Doc. Math. **6** (2001), 447–481 (electronic).

[ST02] ———, *Locally analytic distributions and $p$-adic representation theory, with applications to* $\mathrm{GL}_2$, J. Amer. Math. Soc. **15** (2002), no. 2, 443–468 (electronic).

[ST03] ———, *Algebras of $p$-adic distributions and admissible representations*, Invent. Math. **153** (2003), no. 1, 145–196.

[ST05] ———, *Duality for admissible locally analytic representations*, Represent. Theory **9** (2005), 297–326 (electronic). MR MR2133762 (2006a:22016)

[Tat67] J. T. Tate, *$p$-divisible groups.*, Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin, 1967, pp. 158–183. MR MR0231827 (38 #155)

[Tei] J. Teitelbaum, *Admissible analytic representations. An introduction and three questions.*, Talk at Harvard Eigensemester 2006. Available at: http://www2.math.uic.edu/ jeremy/harvard.

[Tem04] M. Temkin, *On local properties of non-Archimedean analytic spaces. II*, Israel J. Math. **140** (2004), 1–27.

[VdB86] M. Van den Bergh, *A note on graded $K$-theory*, Comm. Algebra **14** (1986), no. 8, 1561–1564.

[Wei] C. Weibel, *An introduction to algebraic K-theory*, Preprint. Available at: http://www.math.rutgers.edu/ weibel/Kbook.html.

[Wei94] ———, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany

*E-mail address*: toschmid@math.uni-muenster.de